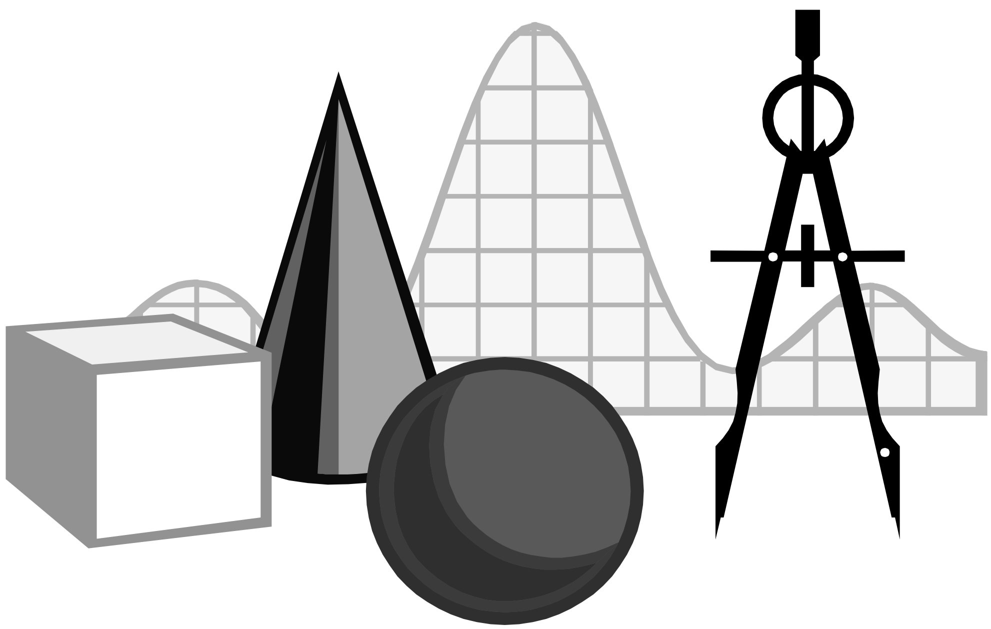
MATHEMATICS STANDARDS OF LEARNING ENHANCED SCOPE AND SEQUENCE

Algebra II



Commonwealth of Virginia
Department of Education
Richmond, Virginia
2004

Superintendent of Public Instruction
Jo Lynne DeMary

Assistant Superintendent for Instruction
Patricia I. Wright

Office of Elementary Instructional Services
Linda M. Poorbaugh, Director
Karen W. Grass, Mathematics Specialist

Office of Middle Instructional Services
James C. Firebaugh, Director

Office of Secondary Instructional Services
Maureen B. Hijar, Director
Deborah Kiger Lyman, Mathematics Specialist

Edited, designed, and produced by the CTE Resource Center
Margaret L. Watson, Administrative Coordinator
Bruce B. Stevens, Writer/Editor
Richmond Medical Park
2002 Bremo Road, Lower Level
Richmond, Virginia 23226
Phone: 804-673-3778
Fax: 804-673-3798
Web site: http://CTEresource.org

The CTE Resource Center is a Virginia Department of Education grant project administered by the Henrico County Public Schools.

NOTICE TO THE READER

In accordance with the requirements of the Civil Rights Act and other federal and state laws and regulations, this document has been reviewed to ensure that it does not reflect stereotypes based on sex, race, or national origin.

The Virginia Department of Education does not unlawfully discriminate on the basis of sex, race, age, color, religion, handicapping conditions, or national origin in employment or in its educational programs and activities.

The content contained in this document is supported in whole or in part by the U.S. Department of Education. However, the opinions expressed herein do not necessarily reflect the position or policy of the U.S. Department of Education, and no official endorsement by the U.S. Department of Education should be inferred.

Introduction

The *Mathematics Standards of Learning Enhanced Scope and Sequence* is a resource intended to help teachers align their classroom instruction with the Mathematics Standards of Learning that were adopted by the Board of Education in October 2001. The Mathematics Enhanced Scope and Sequence is organized by topics from the original Scope and Sequence document and includes the content of the Standards of Learning and the essential knowledge and skills from the Curriculum Framework. In addition, the Enhanced Scope and Sequence provides teachers with sample lesson plans that are aligned with the essential knowledge and skills in the Curriculum Framework.

School divisions and teachers can use the Enhanced Scope and Sequence as a resource for developing sound curricular and instructional programs. These materials are intended as examples of how the knowledge and skills might be presented to students in a sequence of lessons that has been aligned with the Standards of Learning. Teachers who use the Enhanced Scope and Sequence should correlate the essential knowledge and skills with available instructional resources as noted in the materials and determine the pacing of instruction as appropriate. This resource is not a complete curriculum and is neither required nor prescriptive, but it can be a valuable instructional tool.

The Enhanced Scope and Sequence contains the following:

- Units organized by topics from the original Mathematics Scope and Sequence
- Essential knowledge and skills from the Mathematics Standards of Learning Curriculum Framework
- Related Standards of Learning
- Sample lesson plans containing
  - Instructional activities
  - Sample assessments
  - Follow-up/extensions
  - Related resources
  - Related released SOL test items.

Acknowledgments

Marcie Alexander
Chesterfield County

Melinda Batalias
Chesterfield County

Susan Birnie
Alexandria City

Rachael Cofer
Mecklenburg County

Elyse Coleman
Spotsylvania County

Rosemarie Coleman
Hopewell City

Sheila Cox
Chesterfield County

Debbie Crawford
Prince William County

Clarence Davis
Longwood University

Karen Dorgan
Mary Baldwin College

Sharon Emerson-Stonnell
Longwood University

Ruben Farley
Virginia Commonwealth University

Vandivere Hodges
Hanover County

Emily Kaiser
Chesterfield County

Alice Koziol
Hampton City

Patrick Lintner
Harrisonburg City

Diane Leighty
Powhatan County

Marguerite Mason
College of William and Mary

Marcella McNeil
Portsmouth City

Judith Moritz
Spotsylvania County

Sandi Murawski
York County

Elizabeth O'Brien
York County

William Parker
Norfolk State University

Lyndsay Porzio
Chesterfield County

Patricia Robertson
Arlington City

Christa Southall
Stafford County

Cindia Stewart
Shenandoah University

Susan Thrift
Spotsylvania County

Maria Timmerman
University of Virginia

Diane Tomlinson
AEL

Linda Vickers
King George County

Karen Watkins
Chesterfield County

Tina Weiner
Roanoke City

Carrie Wolfe
Arlington City

Organizing Topic Rational and Radical Equations

Standards of Learning

AII.1 The student will identify field properties, axioms of equality and inequality, and properties of order that are valid for the set of real numbers and its subsets, complex numbers, and matrices.

AII.2 The student will add, subtract, multiply, divide, and simplify rational expressions, including complex fractions.

AII.3 The student will
a) add, subtract, multiply, divide, and simplify radical expressions containing positive rational numbers and variables and expressions containing rational exponents; and
b) write radical expressions as expressions containing rational exponents and vice versa.

AII.7 The student will solve equations containing rational expressions and equations containing radical expressions algebraically and graphically. Graphing calculators will be used for solving and for confirming the algebraic solutions.

AII.9 The student will find the domain, range, zeros, and inverse of a function; the value of a function for a given element in its domain; and the composition of multiple functions. Functions will include exponential, logarithmic, and those that have domains and ranges that are limited and/or discontinuous. The graphing calculator will be used as a tool to assist in investigation of functions.

AII.17 The student will perform operations on complex numbers and express the results in simplest form. Simplifying results will involve using patterns of the powers of *i*.

AII.20 The student will identify, create, and solve practical problems involving inverse variation and a combination of direct and inverse variations.

Essential understandings, knowledge, and skills	Correlation to textbooks and other instructional materials
Rational Equations	
• State the domain and range of a rational function.	____________
• Identify the vertical and horizontal asymptotes of a rational function.	____________
• Sketch the graph of a rational function.	____________
• Identify a variation as direct, inverse, or joint.	____________
• Solve practical problems involving joint variation.	____________
• Simplify rational algebraic expressions.	____________
• Compare simplifying rational algebraic expressions to simplifying fractions.	____________
• Add, subtract, multiply, and divide rational algebraic expressions, including complex fractions.	____________
• Solve rational equations algebraically and graphically. The graphing calculator will be used as a primary tool for solution and for checking the algebraic solution.	____________

Radical Equations

- Simplify radical expressions with a variety of indices (rational and integral) until:
  - the index, n, is as small as possible ______________________
  - the radicand contains no factors that are the n^{th} powers of an integer or polynomial ______________________
  - the radicand contains no fractions ______________________
  - no radicals appear in the denominator. ______________________
- Rationalize the denominator of a rational expression that contains a radical expression in the denominator. ______________________
- Add, subtract, multiply, and divide radical expressions. ______________________
- Write expressions in radical form as expressions with rational exponents. ______________________
- Evaluate expressions in either exponential or radical form.
- Solve equations containing radical expressions and equations of n^{th} roots algebraically and graphically. ______________________
- Check solutions for extraneous roots. ______________________
- Recognize a complex number as a number that can be written as $a + bi$ where a and b are real numbers and i is the principal square root of –1. ______________________
- Recognize pure imaginary numbers. Simplify square roots with negative arguments. ______________________
- Represent a complex number geometrically in the coordinate plane. ______________________
- Add, subtract, and multiply complex numbers. ______________________
- Compare and contrast adding, subtracting, and multiplying complex numbers with operating on real numbers. ______________________
- Simplify powers of i and generalize the pattern. ______________________
- Simplify rational expressions with complex numbers in the denominator by using complex conjugates. ______________________
- Place the following sets of numbers in a hierarchy: complex numbers, pure imaginary numbers, real numbers, rational and irrational numbers, integers, whole numbers, and natural numbers. Venn diagrams may be used. ______________________

Rational Expressions

Organizing topic	Rational and Radical Equations
Overview	Students explore the graphs of rational functions and practice the arithmetic with rational expressions.
Related Standards of Learning	AII.2, AII.7, AII.9

Objectives

- The student will state the domain and range of a rational function.
- The student will identify the vertical and horizontal asymptotes of a rational function.
- The student will sketch the graph of a rational function.
- The student will simplify rational algebraic expressions.
- The student will compare simplifying rational algebraic expressions to simplifying fractions.
- The student will add, subtract, multiply, divide, and simplify rational expressions.

Materials needed

- Graphing calculators
- A copy of each of the two "Rational Expressions" handouts for each student

Instructional activity

1. Place students in pairs to work together. (You may change the pairings as students move to the different sections.) Distribute the handouts.
2. Have a class discussion of each section before proceeding to the next.
3. Students may be chosen to present the problems by random draw.
4. The extension handout has several parts that may be used separately.

Follow-up/extension

- The extension handout includes problems on complex fractions and a beginning look at the graphs of the rational functions (continuity — missing point or infinite, limits, etc.). The teacher may take this as far as desired with a particular class.

Homework

- Journal: Compare and contrast simplifying rational algebraic expressions and simplifying fractions.

Rational Expressions

I. Graph each of the following on the graphing calculator and find a) domain, b) range, c) zeros, and d) quick sketch.

1. $y = \dfrac{x+3}{x-5}$
2. $y = \dfrac{x}{x-2}$
3. $y = \dfrac{5}{x-7}$
4. $y = \dfrac{2}{x^2+1}$
5. $y = \dfrac{x+2}{(x+3)(x+1)}$

Conclusions:

6. In general, when given a rational function, how can the domain be determined algebraically?
7. In general, how can the zeros be determined?
8. Given $y = \dfrac{x+3}{x^2-9}$, find the domain and zeros algebraically, and verify with the calculator.

II. Find the domain for each of the following:

9. $y = \dfrac{4}{x}$
10. $y = \dfrac{2x-3}{x^2-36}$
11. $y = \dfrac{2x+3}{2x-3}$
12. $y = \dfrac{x^2-x-42}{x-7}$
13. $y = \dfrac{x+3}{x^2-3x-18}$

III. Simplify the following completely:

14. $\dfrac{-13x^2y^2}{39x^3y}$
15. $\dfrac{x^2-3x-4}{4x-16}$
16. $\dfrac{20x^2-32x}{4x}$
17. $\dfrac{x^2-7x+12}{x^2-5x+6}$
18. $\dfrac{x^2-2x-35}{x^2-10x+21}$

IV. Multiply and divide the following as indicated, and give result in simplest form:

19. $\dfrac{8x-32}{x^3+x^2} \bullet \dfrac{x^5-x^4-2x^2}{4x^2-16x+16}$

20. $\dfrac{x^2+2x-3}{3x^2-x-2} \div (x+3)$

21. $\dfrac{40mn}{18r^2} \div \dfrac{25m^2n}{27r^2m}$

22. $\dfrac{x^2+5x-6}{2x^2} \bullet \dfrac{12x}{4x-4y}$

23. $\dfrac{x^2+8x+16}{c^2-6c+9} \div \dfrac{2x+8}{3c-9}$

24. $\dfrac{7x+7y}{y-x} \div \dfrac{21}{4x-4y}$

V. Add and subtract the following as indicated, and simplify answers:

25. $\dfrac{5}{x} + \dfrac{3}{x-1}$

26. $\dfrac{3-4x}{x^2+3x-10} - \dfrac{2}{x+5}$

27. $\dfrac{x}{x^2-x-30} - \dfrac{1}{x+5}$

28. $\dfrac{x}{x^2-9} + \dfrac{3}{x(x-3)}$

29. $\dfrac{3}{x+1} + \dfrac{4}{2x-6} - \dfrac{x^2-5}{x^2-2x-3}$

Rational Expressions: Extension

I. Simplify the following completely:

1. $\dfrac{\dfrac{x+3}{x^2-25}}{\dfrac{x^2-9}{x+5}}$

2. $\dfrac{\dfrac{1}{x}-\dfrac{1}{2x+1}}{\dfrac{4x}{2x+1}}$

3. $\dfrac{\dfrac{2}{4x+12}}{\dfrac{4}{2x+6}+\dfrac{1}{x+3}}$

4. $\dfrac{\dfrac{4}{x-3}+3}{\dfrac{4x-1}{x-3}+4}$

5. $\dfrac{\dfrac{4}{x^2-25}+\dfrac{2}{x+5}}{\dfrac{1}{x+5}+\dfrac{1}{x-5}}$

II. Given $y=\dfrac{x^2-11x+30}{x^2+3x-4}$

6. Express y in completely factored form.
7. What are the exclusions for y (those numbers that are not in the domain of y)? Why?
8. What do these exclusions tell about the graph of y?
9. What are the zeros for y? Why?
10. What do the zeros tell about the graph of y?
11. Would y create a continuous graph? Explain.

III. Given $y=\dfrac{x-3}{x-5}$

12. Complete these tables of values. Do not use the graphing feature of the graphing calculator to answer!

x	2	4	4.5	4.7	4.9	4.99	4.999
y							

x	8	6	5.5	5.2	5.1	5.01	5.001
y							

13. Describe what these values should tell you about the graph of the given function.
14. Use the graphing calculator to verify your response to #13.

IV. Given $y = \dfrac{x-4}{x^2-16}$

15. Determine the exclusions for this function y.
16. Using the same type of exploration as in the above problem, determine what this graph looks like around its exclusions. Verify with the graphing calculator.

V. Create a rational function that has the following characteristics. Verify your equation with the graphing calculator.

17. Exclusions at $x = 6$ and $x = -3$.
18. Exclusions at $x = 4$ and $x = 5$, and zeros at $x = 2$ and $x = -7$

Direct, Inverse, and Joint Variation

Organizing topic Rational and Radical Equations

Overview This activity provides many variation problems in context.

Related Standard of Learning AII.20

Objectives

- The student will identify a variation as direct, inverse, or joint.
- The student will solve practical problems involving joint variation.

Materials needed

- A "Direct, Inverse, and Joint Variation" handout for each student

Instructional activity

1. Distribute the handouts.
2. Have students complete Part I. Keep close track of time, and have students pair up to check.
3. Have students complete Part II and then pair with a different student to check.
4. Have students complete Part III and then pair with a third student to check.
5. Have students complete Part IV as homework.

Follow-up/extension

- Have students create five to ten variation situations different from any seen in class.

Homework

- Part IV of the handout

Direct, Inverse, and Joint Variation

Part I

Identify each of the following as a direct, inverse, or joint variation by filling in the blank.

1. Volume of a gas, V, at constant temperature varies ____________ with its pressure, P.
2. Intensity of sound varies ___________ with distance away from the object creating the sound.
3. The weight of a body varies ____________ with the square of the distance it is from the center of the earth.
4. The power of an electrical circuit varies ____________ as the resistance and current.
5. The heat loss through a glass window of a house on a cold day varies ____________ as the difference between the inside and outside temperatures and the area of the window, and it varies ____________ as the thickness of the window glass.
6. The amount of sales tax paid varies ____________ as the total of the goods purchased.
7. The time to complete a job varies ____________ as the number of workers working.
8. To balance a seesaw, the distance a person is from the pivot is ____________ proportional to his/her weight.
9. The intensity of a light varies ____________as the square of the distance from the light source.
10. The time it takes to complete a specific trip varies ____________ as the speed of travel.
11. The cost of gas on a trip varies ____________ with the length of the trip.
12. The length of a spring varies ____________ with the force applied to it.
13. The number of congruent marbles that fit into a box is ____________ proportional to the cube of the radius of each marble.
14. The number of people invited to dinner varies ___________ as the amount of space each guest has at the table.
15. The number of people invited to dinner varies ____________ as the number of pieces of silverware used.
16. The time it takes to harvest a crop varies ____________ with the number of people assisting in the harvest.
17. The time it takes a runner to complete a lap on the track varies ____________ as the speed of the runner.
18. The cost of a cake varies ____________as the cake's thickness and the square of the radius.
19. The number of calories burned during exercise varies ____________ with the time spent performing the exercise.
20. The power generated by a windmill is ____________ proportional to the cube of the wind speed.

Part II

Write the equation being described by each:

21. The volume, V, of a balloon is directly proportional to the cube of the balloon's radius, R.
    ____
22. The number, N, of grapefruit that can fit into a box is inversely proportional to the cube of the diameter, D, of each grapefruit. _______________________
23. The time, T, that a plane spends on the runway varies inversely as the take-off speed.
    ____

24. The weight, W, that a column of a bridge can support varies directly as the fourth power of its diameter, D, and inversely as the square of its length, L. __________________________
25. The radiation, R, from the decay of plutonium is directly proportional to the mass, M, of the sample tested and inversely proportional to the square of the distance, D, from the detector to the sample. ______________________________
26. #20 above _____________
27. # 3 above _____________
28. #8 above _____________
29. #9 above _____________

Part III

Write an equation for and solve each of the following:

30. The cost, C, in cents of lighting a 100-watt bulb varies directly as the time, T, in hours, that the light is on. The cost of using the light for 1,000 hours is \$0.15. Determine the cost of using the light for 2,400 hours. ____
31. The power, P, in watts of an electrical circuit varies jointly as the resistance, R, and the square of the current, C. For a 240-watt refrigerator that draws a current of 2 amperes, the resistance is 60 ohms. What is the resistance of a 600-watt microwave oven that draws a current of 5 amperes?____
32. The force needed to keep a car from skidding on a curve varies directly as the weight of the car and the square of the speed and inversely as the radius of the curve. Suppose a 3,960 lb. force is required to keep a 2,200 lb. car, traveling at 30 mph, from skidding on a curve of radius 500 ft. How much force is required to keep a 3,000 lb. car, traveling at 45 mph, from skidding on a curve of radius 400 ft.? ____

Part IV

Write a general equation for each of the following relationships, and sketch it:

33. Y varies directly as X.

34. Y varies directly as the square of X.

35. Y varies inversely as X.

36. Y varies inversely as the square of X.

Rational Equations

Organizing topic Rational and Radical Equations

Overview Students practice solving equations involving rational expressions and applications.

Related Standard of Learning AII.7

Objective

- The student will solve rational equations algebraically and graphically. The graphing calculator will be used as a primary tool for solution and for checking the algebraic solution.

Materials needed

- A "Rational Equations" handout for each student
- Graphing calculators

Instructional activity

1. Place students in groups of no more than three.
2. Have students record all work on their own papers.
3. Work through and then discuss each problem, one at a time, letting student groups explain to the class.
4. You may want to determine which of the problems from the handout will be completed during class, leaving the remaining problems for homework.

Rational Equations

I. Give the exclusions for each of the following rational expressions.

1. $\dfrac{2x+5}{x-7}$
2. $\dfrac{x-5}{x^2-25}$
3. $\dfrac{x^2+5x+6}{x^2+x-2}$
4. $\dfrac{x^2-2x-15}{x^3-4x^2-5x}$
5. $\dfrac{\frac{x-7}{x+2}}{\frac{x}{x+5}}$

II. Solve each of the following. Be sure to identify exclusions and check answers in the original equation.

1. $\dfrac{4}{x}+\dfrac{1}{3x}=9$
2. $\dfrac{3}{n+1}=\dfrac{5}{n-3}$
3. $\dfrac{2}{x+5}-\dfrac{3}{x-4}=\dfrac{6}{x}$
4. $\dfrac{1}{x-5}+\dfrac{1}{x-5}=\dfrac{4}{x^2-25}$
5. $\dfrac{6x^2+5x-11}{3x+2}=\dfrac{2x-5}{5}$
6. $\dfrac{3}{x-1}-\dfrac{4}{x-2}=\dfrac{2}{x+1}$
7. $\dfrac{x}{x^2-4x-12}=\dfrac{x+1}{6-x}-\dfrac{x-3}{2+x}$
8. $\dfrac{c+2}{c-5}=\dfrac{7}{c+2}$
9. $\dfrac{x^2-2x-3}{x^2-x-6}-\dfrac{x}{x+2}=\dfrac{5-x}{x-3}$

III. Solve each of the following, showing all work:

10. A number minus 5 times its reciprocal is 7. Find the number.
11. A swimming pool can be filled in 12 hours, using the large pipe alone, and in 18 hours with the small pipe alone.
    a. If both pipes are used, how long will it take to fill the pool?
    b. If the fire department is asked to help out by using its tank that could fill the pool alone in 15 hours, how long will it take to fill the pool if they all are working together?
    c. At the end of the summer a drain is left open and empties the pool in 24 hours. If the pool attendant, who is preparing the pool for opening day, forgets to plug the drain, can the pool be filled? If so, how long will it take? If not, explain why not.

12. John is deciding between two jobs. One promises convenient hours and is closer to his home. The other pays \$2.25 more per hour, and John would earn \$1,000 in 10 hours less than it will take him to earn \$900 at the first job. Find the hourly wage for both jobs.
13. The current in a river was estimated to be 4 mph. A speedboat went downstream 6 miles and came back 6 miles in 15 minutes. What was the average speed of the speedboat in still water? (Hint: Let r be the average speed in mph in still water.)
14. Find all real numbers x, $x + 2$, and $x + 4$ such that the reciprocal of the smallest number is the sum of the reciprocals of the other two. (Are there rational solutions to this problem?)
15. A manufacturing company has a machine that produces 120 parts per hour. When a new machine is delivered that makes twice as many parts per hour, how long will it take the two machines working together to make 120 parts per hour? (Hint: Let x be the time [number of hours] the machines take to make the 120 parts.)

Rational Exponents

Organizing topic Rational and Radical Equations

Overview These verbal relationships allow students to explore and practice interpreting and using rational exponents. Students also convert between symbolic representations.

Related Standard of Learning AII.3

Objective

- The student will simplify radical expressions with a variety of indices (rational and integral) until
  - the index, n, is as small as possible
  - the radicand contains no factors that are the n^{th} powers of an integer or polynomial
  - the radicand contains no fractions
  - no radicals appear in the denominator.
- The student will write expressions in radical form as expressions with rational exponents.
- The student will evaluate expressions in either exponential or radical form.

Materials needed

- A "Rational Exponents" handout for each student
- Graphing calculators

Instructional activity

- Distribute the handout.
- Have students work individually and then explain to the class.

Rational Exponents

1. The volume, V, of a soap bubble is related to its surface area, A, by the formula $V = 0.094A^{3/2}$. What is the surface area of a bubble with volume 5 cm^3?

2. The number of hours, H, that milk stays fresh is a function of the surrounding temperature, T, (in degrees C): $H(T) = 180 * 10^{(-.04T)}$. Predict how long newly pasteurized milk will stay fresh when stored at 8°C.

3. A fast ship's speed, S, (in knots) varies directly as the seventh root of the power, P, (in horsepower) being generated by the engine: $S = 6.492P^{1/7}$. If a ship is traveling at a speed of 25 knots, about how much horsepower is the engine generating?

4. A formula that the police use for finding the speed, S, (in mph) that a car was going from the length, L, (in feet) of its skid marks is $S = \sqrt{5L}$. The *Guinness Book of Records* reports that in 1960, a Jaguar in England had the longest skid mark, which was recorded as 950 ft.
   a. What was its approximate speed?
   b. About how far does a car travel if it skids from 50 mph to a stop?

5. Mike has $500 he wants to save for college. His goal is to find an investment vehicle that would allow his money to double to $1,000 in four years. What compound annual interest rate, R, would make this happen? $A = P(1 + R)^T$, where P is the original principal, R is the annual interest rate, T is the number of years, and A is the total value of the investment.

6. Complete the following table.

Radical Form	**Exponential Form**	**Numerical Value**	**Simplest Radical Form**	**Exponential Equation**
$\sqrt{16}$				
	$(-32)^{3/5}$			
				$81^{3/4} = 27$
	$8^{(-1/3)}$			
			$2\sqrt[3]{5}$	
	$(-54)^{(-2/3)}$			
	$(??)^{(-3/2)}$	7		
			$3\sqrt[4]{5}$	

Let's Get Radical

Organizing topic Rational and Radical Equations

Overview Students use properties of integral exponents to develop properties of rational exponents and radicals.

Related Standard of Learning AII.3

Objectives

- The student will add, subtract, multiply, and divide radical expressions.
- The student will rationalize the denominator of a rational expression that contains a radical expression in the denominator.

Materials needed

- Computer lab or computer with projection device suitable for viewing by all students

Instructional activity

1. Review the properties of integral exponents with students. The iteration of the "Squares Game" that follows may serve as a guide.
2. Lead a discussion that allows students to expand properties of integral exponents to include rational and negative exponents.
3. Have students use properties of exponents to derive properties of radicals applicable when radical expressions are multiplied, divided, or simplified.
4. Have students use properties of exponents to derive methods to add and subtract radical expressions.
5. In the computer lab, have students go to the Drexel University's Math Forum at http://mathforum.org/library/drmath/view/52643.html and read and take notes on the article, "Using Conjugates to Simplify Radicals." Students should discuss the process with a partner and then participate in a whole class discussion.
6. Provide students with practice opportunities.

Squares Game: Properties of Exponents

1. Cut the squares apart.
2. Match equivalent expressions or sentences.
3. You should get a new four-by-four square.

	$6x^{-4}y$			$64m^{-3}$			z^{20}			1	
$(10s)^2$		$12x^3$	$(5x)^3$		$1/(x^{-2})$	$d/16c4$		$4^6 \bullet 4^{-4}$	$1/13y$		$2c^{-4}d$
	$1/13^x$			1/16			$(2^3)^2$			$(1/2)^{-1}$	
	$(2x)^{-3}$			1			$3xy$			$-27y^3$	
2		13^{-x}	$y^4 \bullet y^3$		$(-x)^3$	$4/x^5$		$m \bullet m$	$3 \bullet 5^3$		$(13y)^{-1}$
	$5/yz^2$			4			1/625			$(-8)^0$	
	$5y^{-1}z^{-2}$			$(w^7)^3$			m^2			36	
1		4^{-2}	$(t2)6$		$1/(2x)^{-4}$	16		$(2x)^2/x^2$	729		$(5^{-2})^2$
	$6^2 \bullet 6^3$			$(-3y)^3$			$(-x)(-x)^2$			$(z^4)^5$	
	x^4			30			$100s^2$			$[(-2)^3]^2$	
$4z^3$		y^7	$16x^4$		$(3^{-2})^{-3}$	$8x$		64	$8^{-1} \bullet 8^1$		$1/8x^3$
	$8xy^2$			73^0			$(-x)^4$			64	

Extraneous Whats?

Organizing topic Rational and Radical Equations

Overview Students discover and apply a powers property to the solving of equations with rational exponents or radical expressions in the equation.

Related Standard of Learning AII.7

Objectives

- The student will solve equations containing radical expressions and equations of n^{th} roots algebraically and graphically.
- The student will check solutions for extraneous roots.

Materials needed

- Graphing calculators

Instructional activity

1. Lead a class discussion with the students in small groups.
2. Ask the students, "Given $a = b$, does it necessarily follow that $a^n = b^n$?" Have them explain their thinking. "If $a^n = b^n$, does it necessarily follow that $a = b$?" Explain.
3. Have the students apply the powers property to equation solving. Use equations containing rational exponents as well as equations containing radical expressions.
4. Have the students check algebraic solutions graphically.
5. Have the students check solutions for extraneous solutions.

Follow-up/extension

- Journal: Why do some equations have extraneous solutions?

Imagine That!

(From *Problems with a Point.* February 20, 2001. ©EDC 2000)

Organizing topic	Rational and Radical Equations
Overview	Students use the Internet to define terms and operations with complex numbers.
Related Standards of Learning	AII.1, AII.17

Objectives

- The student will recognize a complex number as a number that can be written as $a + bi$ where a and b are real numbers and i is the principal square root of –1.
- The student will recognize pure imaginary numbers.
- The student will simplify square roots with negative arguments.
- The student will add, subtract, and multiply complex numbers.
- The student will simplify powers of i and generalize the pattern.
- The student will compare and contrast adding, subtracting, and multiplying complex numbers with operating on real numbers.
- The student will simplify rational expressions with complex numbers in the denominator by using complex conjugates.

Materials needed

- A computer lab with one computer for every two students, *or*
- A single computer and a projection system so that the entire class can see the screen

Instructional activity 1

Present the following to the students:

1. Find two numbers whose sum is 10 and product is 24.
2. Now suppose you want to find two numbers whose sum is 10 and product is 20. Both numbers have a square root involved, as no rational numbers will fit this description. Rational numbers can be written as p/q, where p and q are integers and $q \neq 0$.
   a. Write an equation to help you find these numbers.
   b. Solve your equation to find the two numbers.
   c. Check the result: Do your numbers have a sum of 10? Do they have a product of 20?
3. Even using square roots is sometimes not enough to solve problems of this type.
   a. Try to find two numbers whose sum is 10 and product is 30, using the method you used in problem 2 above.
   b. Why would someone say problem 3.a has no solution?
4. In order to solve problems like problem 2, the number system of rational numbers had to be expanded to include square roots of all positive rational numbers. Suppose you could invent a new number that would allow you to find two numbers whose sum is 10 and whose product is 30.
   a. What number would you invent?
   b. Find the two numbers.
   c. Can you show that your two numbers have a sum of 10 and a product of 30?

5. You may recall that if a and b are non-negative real numbers, $\sqrt{a} \bullet \sqrt{b} = \sqrt{ab}$. Assume that square roots of negative numbers exist. If the property above holds for negative numbers too, then $\sqrt{-9}\sqrt{-4} = \sqrt{(-9)(-4)} = \sqrt{36} = 6$.

6. However, there's a problem. You can also do this:

   $\sqrt{-9}\sqrt{-4} = \sqrt{(9)(-1)}\sqrt{(4)(-1)}$

   $= 3(\sqrt{-1})(2)(\sqrt{-1})$

   $= 6(\sqrt{-1})^2$

   $= 6(-1)$

   $= -6$

   This makes it seem like 6 = –6! Both numbers have 36 as their square, but people have agreed that $\sqrt{a}$ refers only to the non-negative square root of a. That means $\sqrt{36} = 6$, not –6. When you extend the real numbers to include numbers like $\sqrt{-9}$, you want the new numbers to work in the same way as the real numbers. To do that, though, you need to be sure that a calculation like $\sqrt{-9}\sqrt{-4}$ has only one interpretation. The property that $\sqrt{a}\sqrt{b} = \sqrt{ab}$ when a and b are positive is helpful when working with square roots. For example, $\sqrt{54} = \sqrt{(9)(6)} = \sqrt{9}\sqrt{6} = 3\sqrt{6}$. The first step when working with a number like $\sqrt{-9}$ is to separate the 9 from –1, as follows: $\sqrt{-9} = \sqrt{(9)(-1)} = \sqrt{9}\sqrt{-1} = 3\sqrt{-1}$. In fact, all square roots of negative numbers can be separated into a square root of a positive number and $\sqrt{-1}$. That means that introducing only one new number, $\sqrt{-1}$, allows you to find square roots of all negative numbers. People who work with these numbers use the letter i to represent $\sqrt{-1}$.

7. You now have two types of numbers: real numbers and numbers of the form bi, where b is a real number. For these numbers to form a number system, you have to be able to operate on them — to add, subtract, multiply, and divide.

8. For one of the following cases, you won't be able to apply the addition and multiplication rules you already know. Find the result for the other three cases.

   a. $3i + 2i$ b. $3i \bullet 2i$ c. $3 + 2i$ d. $3 \bullet 2i$

9. You now have numbers of three types:

   a. a, where a is a real number

   b. bi, where b is a real number and i is $\sqrt{-1}$.

   c. the combination of a and bi from problem 2 that couldn't be simplified ($a + bi$)

10. The last type actually includes the first two. How can you write –3 and $4i$ in that form?

11. A *complex number* is an expression of the form $a + bi$, where a and b are real numbers and $i^2 = -1$. Once numbers are written in this form, you can add and multiply in the same way you would with real numbers, such as $3 + \sqrt{5}$.

Sample assessment

- Write each complex number in the form $a + bi$, where a and b are real numbers.

  a. $5 + 7\sqrt{-16}$ b. $\sqrt{-16}$ c. 67 d. $\sqrt{16}$

- Expand the following:

  a. $(x + 1)(x - 6)$ b. $(3 + \sqrt{5})(2 - \sqrt{5})$ c. $(1 + 7i)(2 - 2i)$

Instructional activity 2

1. Have the students go to the Math Forum Web site (sponsored by Drexel University) at http://mathforum.org/dr.math. Dr. Math answers questions for anyone who cares to submit.
2. Have them enter in the search window "Imaginary Numbers" and click the **Search** button.
3. Have the students read the article about "What is an imaginary number: What is *i*?" and take notes.
4. Near the end of the article are references to five additional articles about imaginary and complex numbers. Have the students read and note the articles entitled "The Imaginary Number" and "What Are Imaginary Numbers?"
5. Have the students discuss with a partner what they have read and noted, and then have the partners discuss their findings with another pair, considering the following:

   Given that $i = \sqrt{-1}$, find i^2.
   Sample: $i^2 = i \bullet i$
   $i^2 = \sqrt{-1} \bullet \sqrt{-1}$
   $i^2 = -1$
   Have the students complete the following table:

i	i	i^5		i^9	
i^2	-1	i^6		i^{10}	
i^3		i^7		i^{11}	
i^4		i^8		i^{12}	

Sample assessment

- Ask the students, "Do you see a pattern? What is it? How can you use it?"

Instructional activity 2 continued

6. Have the students go back to the Math Forum Web site, read the article on "Complex Numbers: Subtraction, Division," http://mathforum.org/library/drmath/view/53880.html, and take notes on adding, subtracting, and dividing complex numbers.
7. Have the students read and take notes on the following articles:
   - "A Primer on Complex Arithmetic" http://mathforum.org/library/drmath/view/61635.html
   - "Conjugates" http://mathforum.org/library/drmath/view/53423.html
   - "Dividing by Complex Numbers" http://mathforum.org/library/drmath/view/62871.html
8. Provide the students with practice opportunities.

Instructional activity 3

Present the students with the following:

1. By now, you should know how to add, subtract, and multiply complex numbers. You may have noticed that division is not included. The rules for addition, subtraction, and multiplication of real numbers also apply to complex numbers, but the rules for division do not. Division and multiplication are closely related. Since you know how to multiply complex numbers, changing a division problem to a multiplication problem might help. Each division problem below is followed by a partial multiplication problem that has the same result. Fill in the blanks.

   a. $8 \div 2 = 4$ $\qquad$ $8 \bullet \_\_\_\_\_ = 4$

   b. $\frac{7}{3} \div \frac{2}{3} = \frac{7}{2}$ $\qquad$ $\frac{7}{3} \bullet \_\_\_\_ = \frac{7}{2}$

   c. $13 \div -8 = -\frac{13}{8}$ $\qquad$ $13 \bullet \_\_\_\_\_ = -\frac{13}{8}$

   d. $a \div b = \frac{a}{b}$ $\qquad$ $a \bullet \_\_\_\_\_ = \frac{a}{b}$

2. An important concept for rewriting division problems as multiplication problems is that of the *multiplicative inverse.* The product of a number and its multiplicative inverse is 1. How are multiplicative inverses important to the answers to the problems above?

3. Explain how to find the multiplicative inverse of an integer.

4. Now think about multiplicative inverses of complex numbers. Use the method you described in activity 1, problem 4 to find a number whose product with $1 + i$ is 1: $(1 + i)(\_\_\_) = 1$. Is the number you found written in the form $a + bi$?

5. For your answer to be in the form $a + bi$, i can appear only once and in the numerator. To get it in that form, then, you need a way to remove the i term from the denominator.

6. You can rewrite a fraction by multiplying both the numerator and the denominator by the same number. This is the same as multiplying by 1, so the fraction does not change its value.

7. Write the following products as a single fraction. Expand or simplify products when possible. For example: $\frac{3}{10} \bullet \frac{2}{2} = \frac{3 \bullet 2}{10 \bullet 2} = \frac{6}{20}$

   $\frac{2}{9} \bullet \frac{3}{3}$ $\qquad$ $\frac{8}{10} \bullet \frac{\frac{1}{2}}{\frac{1}{2}}$ $\qquad$ $\frac{1}{x+2} \bullet \frac{x+2}{x+2}$ $\qquad$ $\frac{1}{1+i} \bullet \frac{3+2i}{3+2i}$ $\qquad$ $\frac{1}{1+i} \bullet \frac{2-2i}{2-2i}$

   Look carefully at the problems above. In the third problem, you should still have an x^2 term in the denominator. In the fourth problem, though, the denominator should have an i term but no i^2 term. Explain why. In the fifth problem, the denominator should have no i at all. Explain why.

8. Expand each of the following products. Then find the value of b that would give the product no i term.

   $(4 + 3i)(4 + bi)$ $\qquad$ $(3 - 2i)(2 + bi)$ $\qquad$ $(-7 + 8i)(3 + bi)$ $\qquad$ $(-7 + 8i)(-7 + bi)$

Find the pattern in your answers above. Then, for each product below, use your pattern to find values for both a and b that give the product no term with i in it.

$(1 + i)(a + bi)$ $(14 + 5i)(a + bi)$ $(-2 - 20i)(a + bi)$ $(\frac{2}{3} - i)(a + bi)$

9. Now return to the problem of dividing two complex numbers: $(3 + 2i) \div (8 - 3i)$.

   Use your method for finding a multiplicative inverse to find one for $8 - 3i$.

   Find values for a and b so that the product of $8 - 3i$ and $a + bi$ will have no i term.

   Multiply the numerator and denominator of the multiplicative inverse of $8 - 3i$ by $\frac{a + bi}{a + bi}$, using the values for a and b you found.

   Write the result as a complex number: ___ + ___ i.

   Rewrite the division problem as a multiplication problem of two complex numbers, one of which is the multiplicative inverse of $8 - 3i$.

   Find the quotient $(3 + 2i) \div (8 \bullet 3i)$ by expanding and simplifying the equivalent multiplication problem. Write the result in the form $a + bi$.

Sample assessment

- Find the following quotients. Write each result in the form $a + bi$.

  $(-2 - 3i) \div (-10 + 8i)$ $(c + di) \div (r + si)$

Follow-up/extension

1. Using complex numbers, you can find numbers with any sum and any product.
   a. Find two complex numbers whose sum is 10 and product is 30.
   b. Find two numbers whose sum is s and product is p, where s and p are real numbers.
   c. For what values of s and p are the numbers real?
   d. For what values of s and p are the numbers complex?
2. Demonstrate that 13 is not prime, using imaginary numbers. Demonstrate that 3 is prime, using imaginary numbers.

Homework

- Journal: Compare and contrast adding, subtracting, and multiplying complex numbers with operating on real numbers.

Complex Numbers

Organizing topic Rational and Radical Equations

Overview This activity provides practice in complex number arithmetic and a review of field properties.

Related Standard of Learning AII.1

Objectives

- The student will identify field properties for complex numbers.
- The student will place the following sets of numbers in a hierarchy, using a Venn diagram:
  - complex numbers
  - pure imaginary numbers
  - real numbers
  - rational numbers
  - irrational numbers
  - integers
  - whole numbers
  - natural numbers.

Materials needed

- A "Complex Numbers" handout for each student
- A "Complex Numbers: Extension" handout for each student (optional)

Instructional activity

1. Pair students.
2. Distribute handouts.
3. Have textbooks and other resources available for students to use to find information about the field properties if they do not remember them. (Do not answer questions, but challenge the students to find answers.)

Follow-up/extension

- Have the students work through the extension handout dealing with an application of complex numbers.

Complex Numbers

Place the following sets of numbers in a hierarchy, using a Venn diagram:

- complex numbers
- pure imaginary numbers
- real numbers
- rational numbers
- irrational numbers
- integers
- whole numbers
- natural numbers.

Simplify completely over *C:*

1. $\sqrt{-50}$
2. $-\sqrt{-18}$
3. i^{15}
4. i^{50}
5. $i^{13} + i^{23} + i^{40}$
6. $(i\sqrt{5})^2$
7. $(3 + 2i) + (9 + 7i) + (-2 - i)$
8. $(5 - 3i) - (-1 + 4i)$
9. $-1i(9 - 2i) - (5 + 7i)$
10. $-i(5 + 2i)$
11. $(3 + i)(2 + i)$
12. $4i(1 + i) + 3(6 - 2i)$
13. $(3 + 5i)(8 - i)$
14. $i(4 - i)(3 + 2i)$
15. $(2 - 3i)^4$
16. $\dfrac{5-i}{i}$
17. $\dfrac{2+i}{2-i}$
18. $\dfrac{3-4i}{2+5i}$

Identify the complex number being described in each of the following, and verify your answer.

- the identity for addition on *C*
- the multiplicative identity on *C*
- the additive inverse of $a + bi$
- the multiplicative inverse of $a + bi$

For each of the following, give the representation of the statement in symbols, and state whether the statement is true or false. If false, give a counterexample; if true, verify (prove) the statement.

- The sum of two imaginary numbers is imaginary.

- The product of two imaginary numbers is real.

- Addition of complex numbers is commutative.

- Multiplication of complex numbers is associative.

- The distributive property of multiplication over addition holds for complex numbers.

- The set of imaginary numbers is closed under addition.

- The set of complex numbers is closed under multiplication.

Complex Numbers: Extension

Complex numbers are commonly used in electronics in dealing with alternating current (AC) representing voltage, current, and impedance.

- **Voltage,** V, (in volts) — the electrical potential between two points in an electrical circuit
- **Current,** I, (in amps) — the rate of flow of electrical charge through a circuit
- **Impedance,** Z, (in ohms) — the opposition to the flow of current caused by resistors, coils, and capacitors.

The total impedance in a circuit is represented by a complex number whose real part denotes the opposition to current flow due to resistors and whose imaginary part represents opposition due to coils and capacitors.

$8 - 3i$ total impedance:

8 ohms — resistors' impedance

3 ohms — coils' and capacitors' impedance

$I = \frac{V}{Z}$ I = current, V = voltage, Z = impedance

The total impedance of two circuits in series (flowing one to the other) is found by adding the impedances of the two individual circuits. The total voltage of two circuits in series is found by adding the voltages of the two individual circuits.

- Suppose two AC circuits are connected in series, one with impedance of $-5 + 7i$ ohms, the other with impedance $8 - 13i$ ohms.
  - Find the total impedance.
  - If the total voltage is 15V, find the current.
- In an AC circuit, V across resistors is 15V, and total V across the coils and capacitors is 18V.
  - Find the total voltage.
  - Find the impedance.
  - If this circuit is really two circuits in series and the impedance of one is $2.5 - 2i$ ohms, find the impedance of the other.

Graphing Complex Numbers

(From *Problems with a Point.* February 20, 2001. ©EDC 2000)

Organizing topic	Rational and Radical Equations
Overview	Students extend graphing real numbers in the Cartesian plane to representing complex numbers in a two-dimensional plane.
Related Standard of Learning	AII.17

Objective

- The student will represent a complex number geometrically in the coordinate plane.

Materials needed

- Graph paper

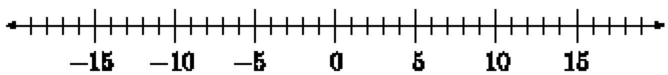
Instructional activity

Present the following to the students:

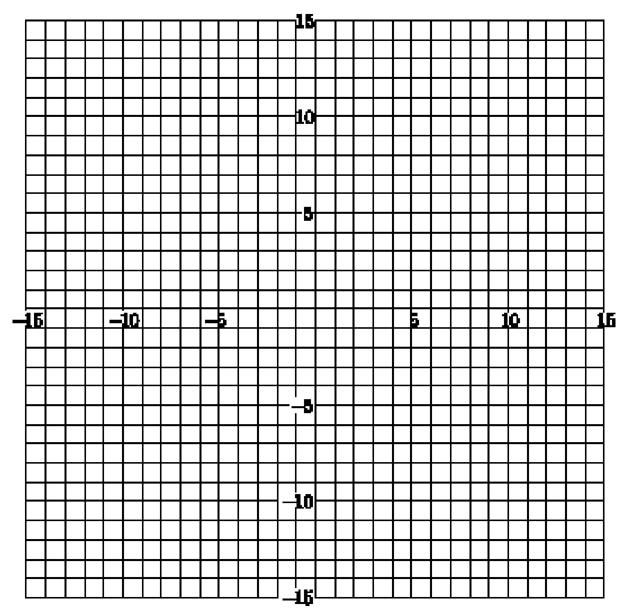
1. The types of numbers you find on a number line are real numbers. As you know, every real number is also a complex number. If a complex number $a + bi$ is also a real number, what do you know about a or b?
2. Real numbers can be graphed as points on a number line. To create a graphic representation of complex numbers, however, a number line is not enough. The form of a complex number has two *parameters*, a and b. The first parameter is the real part of the number, and the second (when multiplied by i) is the imaginary part. That is, the numbers are ordered; so $5 – 2i$ is not the same as $–2 + 5i$.
3. Suppose you want to graph a single object (a point) that has an ordered pair of numbers. What graphing system could you use?
4. Make a rough sketch of a graph showing the locations of $5 – 2i$ and $–2 + 5i$. Provide enough labels on your graph so that someone who has not seen complex numbers on such a graph can figure out which complex numbers are represented by the points.
5. You probably used a coordinate grid to make the sketch in the preceding step. Which number did you use for the horizontal axis, the real part or the imaginary part?
6. The horizontal axis really could have been used for either part, the real or the imaginary. However, the convention used by people who work with a coordinate interpretation of complex numbers is to use the real part for the horizontal axis and the imaginary part for the vertical axis.
7. The grid on which you can plot ordered pairs (x, y), where x and y are real, is sometimes called the *coordinate plane*. The grid on which you can plot complex numbers, $a + bi,$ is called the *complex plane*.
8. Create a grid for complex numbers, and plot the following numbers on it:
   - $1 + i$
   - $3i$
   - $–1 + i$
   - 3
   - $7 – 3i$
   - $0w$

- $-2 - 4i$
- $i - 2$

9. Suppose you have an equation, $ax = b$, where a is nonzero.
   - If a and b are real, can you always find a solution to this equation?
   - If a and b are real, but x must be an integer, can you always find a solution?
   - When a and x are both integers, b has a special relationship to a. What is that relationship?
10. Suppose $a = 3$, so the equation is $3x = b$.
   - On the number line, mark all the values of b for which the equation $3x = b$ has an integral solution.
   - What can you say about the locations of the possible b values?



   - With real numbers, you usually talk about b being a multiple of a only if $ax = b$ and a, b, and x are all integers. The analog to integers in the complex plane are *Gaussian integers* — complex numbers of the form $x + yi$, where x and y are integers.
11. Consider the Gaussian integer $a = 3 + 2i$. For each of the following values of b, decide if the solution to $ax = b$ is a Gaussian integer. If it is, what is the solution?
   - $2 + 3i$
   - $7 + 13i$
   - $1 + 8i$
   - $-3 - 11i$
   - $-14 - 8i$
   - $12 - 5i$
12. Generate additional multiples of $3 - 2i$ by doing the following:
   - Multiply $3 - 2i$ by x (where $x = -2, -1, 0, 1, 2$)
   - Multiply $3 - 2i$ by xi (where $x = -2, -1, 1, 2$)
   - Multiply $3 - 2i$ by $(-2 + i)$, $(-1 + i)$, i, $(1 + i)$, and $(2 + i)$
13. Plot the multiples you created in problem 4 on the complex plane grid at right. Then plot those values from the problem above that were multiples of a. Can you find a pattern in the locations of these values?



Follow-up/extension

- Plot some more multiples of a, as needed, then form a conjecture about the locations of the multiples of $3 - 2i$ on the complex plane. Prove your conjecture.

Homework

- Generalize your conjecture for multiples of any Gaussian integer $c + di$. Prove the generalized conjecture.

Sample resources

Mathematics SOL Curriculum Framework
http://www.pen.k12.va.us/VDOE/Instruction/Math/math_framework.html

SOL Test Blueprints

Released SOL Test Items http://www.pen.k12.va.us/VDOE/Assessment/Release2003/index.html

Virginia Algebra Resource Center http://curry.edschool.virginia.edu/k12/algebra

NASA http://spacelink.nasa.gov/.index.html

The Math Forum http://forum.swarthmore.edu/

4teachers http://www.4teachers.org

Appalachia Educational Laboratory (AEL) http://www.ael.org/pnp/index.htm

Eisenhower National Clearinghouse http://www.enc.org/

Sample assessment

Which is equivalent to $2\sqrt{12}+3\sqrt{3}$?

F $16^{\frac{1}{2}}$

G $5\sqrt{5}$

H $7\sqrt{3}$

J 7

The time it takes to travel a given distance varies inversely as the average rate of travel. Averaging 42 miles per hour, it takes Andre 5 hours to drive to Donnegal. If it took him 4 hours and 20 minutes to reach Donnegal on his last trip, what was his average rate of travel?

A 36.4 mph

B 46.7 mph

C 48.5 mph

D 49.4 mph

Which is equivalent to $(4 - 3i)^2$?

A 25

B $25 - 2i$

C 7

D $7 - 24i$

What number does i^4 equal?

F i

G -1

H $-i$

J 1

What is the solution to $\frac{x}{2x+1}=\frac{4}{3}$?

F $x=-\frac{1}{5}$

G $x=-5$

H $x=-\frac{4}{5}$

J $x=-\frac{5}{4}$

What is the solution to $\frac{6a+12}{a}\bullet\frac{a^3}{a+2}$?

F $6a^2$

G $\frac{a^2}{6}$

H $\frac{6(a+2)}{a}$

J $\frac{6a^2+24a+24}{a^4}$

Which is equivalent to $\frac{3x}{7}+\frac{5y}{14x}$?

A $\frac{8y}{21}$

B $\frac{x^2}{14}$

C $\frac{6x^2+5y}{14x}$

D $\frac{3x^2+5y}{14x}$

Which is equivalent to $16^{\frac{3}{4}}$?

F 4

G 8

H 12

J 32

Which is equivalent to $a^{\frac{1}{2}} + b^{\frac{3}{4}}$?

A ab^3

B $\sqrt{ab^3}$

C $\sqrt[3]{a^2b^4}$

D $\sqrt[4]{a^2b^3}$

Organizing Topic Systems of Equations and Inequalities

Standards of Learning

AII.1 The student will identify field properties, axioms of equality and inequality, and properties of order that are valid for the set of real numbers and its subsets, complex numbers, and matrices.

AII.11 The student will use matrix multiplication to solve practical problems. Graphing calculators or computer programs with matrix capabilities will be used to find the product.

AII.12 The student will represent problem situations with a system of linear equations and solve the system, using the inverse matrix method. Graphing calculators or computer programs with matrix capability will be used to perform computations.

AII.13 The student will solve practical problems, using systems of linear inequalities and linear programming, and describe the results both orally and in writing. A graphing calculator will be used to facilitate solutions to linear programming problems.

AII.14 The student will solve nonlinear systems of equations, including linear-quadratic and quadratic-quadratic, algebraically and graphically. The graphing calculator will be used as a tool to visualize graphs and predict the number of solutions.

Essential understandings, knowledge, and skills

Correlation to textbooks and other instructional materials

Matrices

- Organize data into matrices and identify the dimensions of the matrix.
- Investigate commutativity and associativity of matrix addition. Compare and contrast matrix addition with addition of real numbers.
- Multiply matrices using a calculator or a computer with matrix capability.
- Solve problems that require matrix multiplication.
- Investigate commutativity and associativity of matrix multiplication. Compare and contrast matrix multiplication with multiplication of real numbers.
- Find the determinant of a square matrix.
- Identify the identity matrix (I).
- For a matrix A, find the inverse matrix A^{-1} (if it exists) such that $A * A^{-1} = A^{-1} * A = I$.
- Use an inverse matrix to solve matrix equations. Use the graphing calculator or a computer application with matrix capabilities.
- Compare and contrast solving matrix equations and linear equations.

- Represent a system of equations as a matrix equation where the coefficient matrix times the variable matrix equals the constant matrix. ____________________________
- Solve systems of linear equations using inverse matrices. Use the graphing calculator or a computer application with matrix capabilities. ____________________________

Linear Inequalities

- Solve a system of linear inequalities by graphing. ____________________________
- Identify examples of the properties of inequality and order that occur while solving inequalities. ____________________________
- Find the maximum and minimum values of a function over a region (linear programming). ____________________________
- Identify the constraints in a practical situation and model them as inequalities. ____________________________
- Graph the system of inequalities and identify the area of intersection as the feasible region. The feasible region contains all solutions possible. ____________________________
- The maximum and minimum values of the function occur at the vertices of the feasible region. Substitute the coordinates of each vertex of the feasible region into the function to determine which vertex yields the maximum (or minimum) value of the function. ____________________________
- Describe the results of a linear programming problem orally and in writing. ____________________________

Nonlinear Systems

- Solve linear-quadratic systems of equations algebraically and identify the set of ordered pairs that is the solution to the system. ____________________________
- Solve linear-quadratic systems of equations graphically and identify the set of ordered pairs that is the solution to the system. ____________________________
- Solve quadratic-quadratic systems of equations algebraically and identify the set of ordered pairs that is the solution to the system. ____________________________

A Practical Problem to Begin Matrices

Organizing topic	Systems of Equations and Inequalities
Overview	This activity provides a context within which students can practice and apply using matrices to solve a practical problem.
Related Standard of Learning	AII.12

Objective

- The student will represent information and solve for variables, using matrices.

Materials needed

- Graphing calculators
- A "Beginning Matrices" handout for each student

Instructional activity

- Have students work alone or in pairs on this activity.
- Have students explain what the dimensions of a matrix are and the notation associated with matrices.

Beginning Matrices

1. Place the following information into matrices, which will help organize the data.

   Gertrude, Marilda and Homer love to eat at fast food places. Generally,
   - Gertrude orders 4 hamburgers, 4 sodas, and 4 French fries
   - Marilda orders 6 hamburgers, 1 soda, and 1 French fry
   - Homer orders 2 hamburgers, 4 sodas, and 2 French fries.

   They shop around and find the following information for their favorite fast food places:
   - Wendy's charges $1.69 for hamburgers, $.79 for sodas, and $.69 for French fries.
   - McDonald's charges $1.85 for hamburgers, $.59 for sodas, and $.75 for French fries.
   - Burger King charges $1.75 for hamburgers, $.65 for sodas, and $.59 for French fries.

2. Use the matrices to determine how much each person will spend at each fast food place.

3. Recommend a fast food place for each person. Explain your choices.

Operations with Matrices

Organizing topic Systems of Equations and Inequalities

Overview Students review adding and subtracting matrices and expand those ideas to multiplying matrices.

Related Standards of Learning AII.1, AII.11

Objectives

- The student will organize data in matrices and identify the dimensions of the matrix.
- The student will investigate commutativity and associativity of matrix addition.
- The student will compare and contrast matrix addition with addition of real numbers.
- The student will multiply matrices, using a calculator or a computer with matrix capability.
- The student will solve problems that require matrix multiplication.
- The student will investigate commutativity and associativity of matrix multiplication.
- The student will compare and contrast matrix multiplication with multiplication of real numbers.

Materials needed

- Graphing calculators or computers with matrix capability
- Computer lab or computer connected to projection device that all students can see

Instructional activity

1. Give students two or three problems involving adding and subtracting matrices. Review the dimensions of a matrix if necessary and matrix notation. Ask students if matrix addition is commutative. Ask for a demonstration and then for a proof. Allow students to work together in small groups.
2. Have students go to http://www.sosmath.com/matrix/matrix1/matrix1.html, read, and note the article entitled "Multiplication of Matrices." Alternatively, have them read from their textbooks about matrix multiplication. Give students two problems in matrix multiplication, and have them apply the information from the article or text.

Follow-up/extension

- Have the students answer the following: Is matrix multiplication commutative? Associative?
- Have the students prove their answers.

Homework

- Journal: Compare and contrast matrix addition and multiplication with addition and multiplication of real numbers.

Matrix Multiplication

Organizing topic Systems of Equations and Inequalities

Overview This activity allows students to practice and apply matrix multiplication.

Related Standard of Learning AII.11

Objective

- The student will solve problems that require matrix multiplication.

Materials needed

- Graphing calculators
- A “Matrix Multiplication” handout for each student

Instructional activity

1. Use this activity directly following instruction on multiplying matrices.
2. Distribute the handouts.
3. Have students work individually on problems 1 through 6 and then pair up to check.
4. Allow students to work with a partner through the remaining problems on the handout.

Matrix Multiplication

Part I. Matrix Multiplication

Complete the following six matrix multiplication problems by hand, working individually. Once you have completed these problems, find another student in class who has also finished and compare results. Be sure that you and your partner fully discuss any errors before changing your work. Correct any of your errors, using a different color.

For the following matrices, find the product, if it is defined. If the product cannot be accomplished fully, explain why.

1. $\begin{pmatrix} 3 & 4 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \end{pmatrix}$

2. $\begin{pmatrix} 2 & -3 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ 7 & 5 \end{pmatrix}$

3. $\begin{pmatrix} 3 & -2 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \\ -7 \end{pmatrix}$

4. $\begin{pmatrix} -7 \\ 4 \\ 6 \end{pmatrix} \begin{pmatrix} -5 & 3 & -4 \end{pmatrix}$

5. $\begin{pmatrix} 3 & -1 & 4 & 2 \\ 2 & 1 & 0 & 7 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 4 & 6 \\ -7 & 3 \end{pmatrix}$

6. $\begin{pmatrix} -2 & 1 \\ 4 & 3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 3 & -7 & 4 & 1 \\ 5 & 0 & -4 & -3 \end{pmatrix}$

Part II. Snowboard Production

A factory located in Colorado produces snowboards. Each one requires 3 hours of fabricating (assembly) work and 1.5 hours in a finishing process. The fabricating workers earn $12 per hour; the finishing worker $10.

7. Create a row matrix to represent the time it takes to compete a snowboard.
8. Create a column matrix containing the wage data (be careful of the order of these elements).
9. The product of these two matrices will represent the total cost involved in the production of a snowboard. Find this product.
10. This factory also produces "trick" snowboards that are designed specifically for jumps, turns, flips, etc. These trick boards require 5 hours in the fabricating process and 2.5 hours in finishing. Find the total labor cost for producing one of these boards, using the product of the appropriate row and column matrices.
11. By combining the time information from #7 and #8, create a single matrix, called *T*. Let each row of *T* represent the information for the two different boards and each column the different parts of the production process.

	Hours in Fabricating	Hours in Finishing	
Snowboard	__________	__________	= *T*
Trickboard	__________	__________	

12. In addition to the Colorado plant, the factory also has another major production plant in Michigan. Both types of boards are produced here, take the same amount of time, but labor costs differ. In Michigan the fabricating workers earn \$14.50 per hour and the finishers make \$16.50. Create a matrix called R that displays the data about hourly wage rates for both locations and both departments.

	Colorado	Michigan	
Fabricating	________	________	= R
Finishing	________	________	

13. Find the product TR.
14. Find the product RT.
15. Note that both of the products TR and RT are mathematically defined and possible. In the given context, however, only one of these products, TR or RT, has any meaning. Which product has meaning? Thoroughly explain why your choice has meaning and why the other product does not make sense in this context.
16. The company now needs to determine the total hours that need to be scheduled in each processing department (fabricating and finishing) in order to produce 1,200 snowboards and 800 trickboards. Express these production requirements in a matrix called P.
17. Use this matrix P and the time matrix T to determine the total hours needed.

Homework/extension

In the set of real numbers, if $a * b = 0$, then either a or b must be zero. The zero matrix is a matrix whose elements are all zeros. In matrix multiplication, $A * B = 0$ does not mean that one of the matrices A or B must be a zero matrix.

1. Find two non-zero matrices, at least 2 by 2, whose product is zero. (Hint: Look for a smaller dimension matrix first). Remember that a matrix that has only one non-zero entry is not the zero matrix.

Given the matrices

$$A = \begin{pmatrix} 2 & 5 & -6 \\ 1 & 4 & 7 \end{pmatrix} \quad B = \begin{pmatrix} -3 & 5 \\ 6 & -4 \end{pmatrix} \quad C = \begin{pmatrix} 4 & -5 & 2 \\ -1 & 6 & 0 \\ 3 & 4 & -2 \end{pmatrix} \quad D = \begin{pmatrix} -2 & 5 \\ 9 & 4 \\ -7 & 3 \end{pmatrix}$$

Find, if possible

2. AB
3. BA
4. AC
5. DA
6. B^2
7. D^2
8. $-2D + CD$
9. $4BA - 2AC$
10. BAD

Square Patio Patterns

Organizing topic	Systems of Equations and Inequalities
Overview	Students work through a context to represent data in matrix form and use matrix multiplication.
Related Standard of Learning	AII.11

Objective

- The student will solve problems that require matrix multiplication.

Materials needed

- Corners: marshmallows
- Border stabilizers: colored toothpicks
- Frames: wooden toothpicks
- Tiles: squares cut to the length of the wooden toothpicks
- Graphing calculator and view screen
- Overhead projector
- Graphing calculator for each student
- A "Square Patio Patterns" handout for each student

Instructional activity

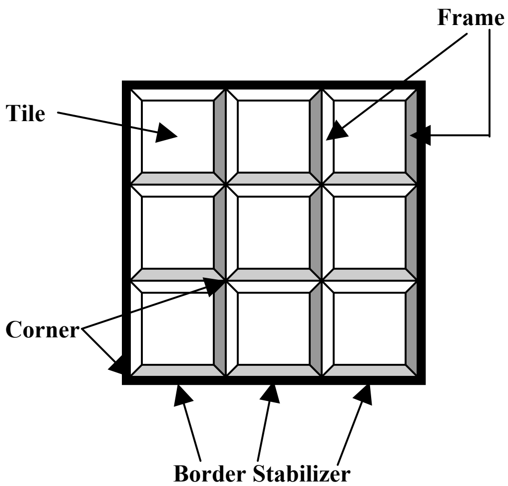
1. This is a group activity lasting at least two days — one day to construct the square patios and collect the data and one day to analyze the data.
2. This activity may be revisited many times.

Follow-up/extension

Matrix Multiplication

- Have students use matrix multiplication to determine the cost of building a real "square patio" of any size. All that is needed for this are prices for actual tiles, corners, border stabilizers, and frames. The teacher may determine these prices, or students may go to a hardware store and collect the prices.

Square Patio Patterns



Patio Dimension	Tiles Needed	Frames Needed	Corners Needed	Border Stabilizers Needed
1-by-1				
2-by-2				
3-by-3				
4-by-4				
5-by-5				
6-by-6				
7-by-7				
8-by-8				

THINK #1

1. Build square patios, and record your data in the chart above.
2. Do you "see" any patterns emerging?
3. Is it easy to predict the number of tiles needed for the different sized patios?
4. How many tiles would you need for a 10-by-10 patio?
5. How many tiles would you need for a 20-by-20 patio?
6. How many tiles would you need for an *n*-by-*n* patio?
7. Do you know any "mathematical" vocabulary that describes the number of tiles needed for each sized patio?

THINK #2

Sometimes there is so much data that we cannot find patterns easily. We must look in logical places for patterns. Sometimes we "stumble across" the patterns.

1. On what does each piece of data *depend*? In other words, does the number of border stabilizers *depend* on the number of corners or the size of the patio?
2. Can you find a pattern that describes the relationship between the size of the patio and the number of border stabilizers needed?

3. Do you know any “mathematical” vocabulary that describes the number of border stabilizers for each sized patio?
4. Can you write a “formula” to describe the relationship between patio size and number of border stabilizers that is true no matter how large the patio is?

THINK #3

1. Does it seem reasonable that the number of corners and the number of frames also depend on the patio size?
2. Can you find a relationship between the patio size and the number of frames? (You may want to consider looking at the number of tiles for a hint.)
3. Can you find a relationship between the patio size and the number of corners? (You may need to draw a few pictures first to see this one.)

Calculating Population Estimates

(From *Problems with a Point.* February 1, 2001. ©EDC 2000)

Organizing topic Systems of Equations and Inequalities

Overview Students work with population information organized into matrices and solve problems.

Related Standard of Learning AII.11

Objective

- The student will solve problems that require matrix multiplication.

Instructional activity

The following table shows information about population change in the United States from 1993 to 1998. Birth and death rates are percentages. Numbers for population are approximate, and the numbers for both population and immigrants are given in thousands.

Year	Population	Birth Rate	Death Rate	Number of Immigrants
1993	257,796	1.55	0.88	880.014
1994	260,292	1.52	0.88	804.416
1995	262,890	1.48	0.88	720.461
1996	265,180	1.47	0.87	915.900
1997	267,636	1.45	0.86	798.378
1998	270,029	1.46	0.87	660.477

Sources: National Center for Health Statistics; Immigration and Naturalization Service

Using matrices can help you make calculations with the information in the table.

1. The birth rate is given as a percentage of the population. To find the number of births in a year, you must multiply the population for that year by the decimal form of the birth rate. Use the table to find the total number of births from 1993 to 1998.
2. Examine these two matrices:

$$\begin{bmatrix} 257,796 & 260,292 & 262,890 & 265,180 & 267,636 & 270,029 \end{bmatrix}$$

$$\begin{bmatrix} 0.0155 & 0.0088 \\ 0.0152 & 0.0088 \\ 0.0148 & 0.0088 \\ 0.0147 & 0.0087 \\ 0.0145 & 0.0086 \\ 0.0146 & 0.0087 \end{bmatrix}$$

a. What information is given in each matrix? Explain what data is included in each column and row.

b. Multiply the matrices, with the first matrix on the left. As you do so, think about the process you go through to find the product — what data is multiplied and added together. Then identify the information given by the product matrix.

c. Compare the process you followed to multiply these matrices with the process you followed in problem 1. How are they similar, and how are they different? Compare the results as well.

3. Now consider the following matrices:

$$B = \begin{bmatrix} 0.0155 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0152 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0148 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0147 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0145 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0146 \end{bmatrix}$$

$$D = \begin{bmatrix} 0.0088 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0088 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0088 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0087 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0086 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0087 \end{bmatrix}$$

$$P = \begin{bmatrix} 257,796 & 260,292 & 262,890 & 265,180 & 267,636 & 270,029 \end{bmatrix}$$

a. What information is given in each matrix?

b. Find the product *PB*, and state what information is given by the product matrix.

c. Find the product *PD*, and state what information is given by the product matrix.

d. Except for the diagonals, every entry in matrices *B* and *D* is 0. What would happen if any of the 0 entries were nonzero?

4. Create a matrix that will allow you to add the number of immigrants for each year to the population. Call the matrix *M*.

5. Using the matrices from problems 2 and 3, write an expression that will give estimated populations for years 1993 to 1998. Calculate your expression.

6. Compare the populations given in the table to the populations you calculated in problem 4.

Follow-up/extension

Fibonacci matrices

1. The *Fibonacci sequence* of numbers starts with 1, 1, and each number after the first two is the sum of the two numbers before it. Hence, the third number is 2, because 1 + 1 = 2, and the fourth number is 3, because 1 + 2 = 3. Then the next would be 5, because 2 + 3 = 5. Find the next four numbers in the sequence:

   1, 1, 2, 3, 5, ___, ___, ___, ___

2. What is the tenth number in the Fibonacci sequence?

3. What is the 20th number in the sequence? (You may use a calculator to find the answer, if you like.) Would it be difficult to find the 50th number?

4. What is a good way to find any number in the sequence? Can this be done without finding all previous numbers in the sequence?
5. Use the matrix sequence and a graphing calculator (or computer algebra system) to find the 50th number in the Fibonacci sequence.

Teacher's notes

- Finding any number in the Fibonacci sequence is easy if you use matrices, as shown below:
  - Define a matrix *A* as follows:

    $$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

  - The first two numbers in the Fibonacci sequence are 1, 1. Write them as a 2 × 1 matrix *B*:

    $$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

  - Find the product *AB*.
  - Multiply the result by *A* on the left — that is, find $A(AB)$, which is A^2B.
  - Now find A^3B.
  - Put the results together to form a sequence of matrices, $B, AB, A^2B, A^3B, \ldots$. Describe the entries of A^nB as specifically as you can.
  - Explain why the entries of A^nB are Fibonacci numbers.

Determinants, Identities, and Inverses

Organizing topic Systems of Equations and Inequalities

Overview Students use the Internet as a resource for instruction on inverse matrices and the identity matrix.

Related Standards of Learning AII.11, AII.12

Objectives

- The student will find the determinant of a square matrix.
- The student will identify the identity matrix.
- The student will, for a matrix A, find the inverse matrix A^{-1} (if it exists).

Materials needed

- Computer lab or computer with projection device suitable for viewing by all students

Instructional activity

1. In the computer lab, have students go to the Math Forum Web site (sponsored by Drexel University) and read and take notes on the following articles:
   - "Explaining the Determinant," http://mathforum.org/library/drmath/view/51439.html
   - "Definition of an Identity Matrix," http://mathforum.org/library/drmath/view/53918.html
   - "Matrix Inverses," http://mathforum.org/library/drmath/view/55484.html
   - "Questions about Matrices," http://mathforum.org/library/drmath/view/55464.html.
2. Have the students discuss the articles in partner pairs.
3. After these discussions, have the students participate in a whole class discussion.
4. Provide the students with practice opportunities.

Balance the Cookies

Organizing topic Systems of Equations and Inequalities

Overview Students solve a system of equations, using matrices and nontraditional methods.

Related Standards of Learning AII.1, AII.12

Objectives

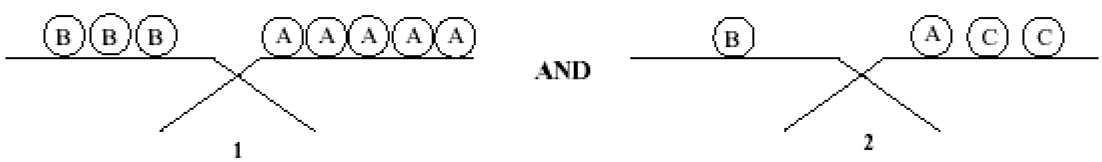
- The student will use an inverse matrix to solve a matrix equation.
- The student will represent a system of equations as a matrix equation where the coefficient matrix times the variable matrix equals the constant matrix.
- The student will solve systems of linear equations, using inverse matrices.

Materials

- Graphing calculators or computers with matrix capabilities

Instructional activity

At the bake sale, cookies are sold by weight. Each apple cookie (A) weighs the same as every other apple cookie. Each banana cookie (B) weighs the same as every other banana cookie but is a different weight than an apple cookie. Each chocolate cookie (C) weighs the same as every other chocolate cookie but is a different weight than either an apple cookie or a banana cookie.
The cookies might be balanced in the following way:



1. How many chocolate cookies will it take to balance one apple cookie?

2. Write equations that reflect the modeling in the pan balances shown in the diagram above.

3. As part of your solution, provide both algebraic and pictorial support.

Fruits

(From *Problems with a Point.* May 23, 2001. ©EDC 2001)

Organizing topic	Systems of Equations and Inequalities
Overview	Students solve a system of equations, using matrices and nontraditional methods.
Related Standards of Learning	AII.1, AII.12

Objectives

- The student will use an inverse matrix to solve a matrix equation.
- The student will represent a system of equations as a matrix equation where the coefficient matrix times the variable matrix equals the constant matrix.
- The student will solve systems of linear equations, using inverse matrices.
- The student will compare and contrast solving matrix equations and linear equations.

Instructional activity

Two bags of apples, a bunch of bananas, and a head of cauliflower cost $8. One bag of apples, two bunches of bananas, and a head of cauliflower cost $10.

1. Find the price of:
    - 3 bags of apples, 3 bunches of bananas, and 2 heads of cauliflower
    - 5 bags of apples, 4 bunches of bananas, and 3 heads of cauliflower
    - 6 bags of apples, 6 bunches of bananas, and 4 heads of cauliflower
    - 5 bags of apples, a bunch of bananas, and 2 heads of cauliflower
    - 3 bunches of bananas and a head of cauliflower
    - 3 bags of apples and a head of cauliflower.
2. Which costs more, a bag of apples or a bunch of bananas?
3. What is the difference in price between a bag of apples and a bunch of bananas?

Homework

- Journal: Compare and contrast solving matrix equations and linear equations.

Using Matrices to Solve Systems of Equations

Organizing topic	Systems of Equations and Inequalities
Overview	Students develop an understanding of the use of matrices in solving systems of equations.
Related Standards of Learning	AII.11, AII.12

Objectives

- The student will use an inverse matrix to solve a matrix equation.
- The student will represent a system of equations as a matrix equation where the coefficient matrix times the variable matrix equals the constant matrix.
- The student will solve systems of linear equations, using inverse matrices.

Materials needed

- Graphing calculators or computers with matrix capability
- Computer lab or computer with projection device suitable for viewing by all students
- A copy of each of the two "Using Matrices to Solve Systems of Equations" handouts for each student

Instructional activity

1. In the computer lab, have students go to the Math Forum at Drexel University http://mathforum.org/library/drmath/view/62883.html and read and note the article, "Inverse Matrix."
2. Divide the students into groups of two or three.
3. Have the groups work through the part I of the handout, with each student recording the group's work.
4. Have each student select his/her own problems to complete for part II, but allow the students to continue to have the support of their group members as they work.

Follow-up/extension

- Vanna is selling vowels. Each of the five, *a, e, i, o,* and *u*, costs a different amount, and you must pay *separately* for each time a vowel appears in a word. For a limited time, she is offering consonants for free! During this special sale, you may purchase the word *audacious* for only $260. *Equivocation* will cost you $340. *Onomatopoeia* sells for $435. *Alliteration* will cost you $325, and *personification* $355. How much does each vowel cost?

Homework

- Have the students choose and complete in detail one of the problems from each set not completed during the group activity.

Using Matrices to Solve Systems of Equations — Part I

Working with members of your group, work through all of the following problems. Be sure to record all the work of your group on your own paper.

Find the following:

1. $\begin{bmatrix} 6 & 4 & -2 & 13 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 3 \\ -9 \end{bmatrix}$

2. $\begin{pmatrix} 4 & 7 & 5 \\ -3 & 2 & -6 \end{pmatrix} \begin{pmatrix} -3 & 6 \\ 5 & -7 \\ 4 & 0 \end{pmatrix}$

Given the following matrix equations, use matrix multiplication to rewrite each of them into a system of equations:

3. $\begin{pmatrix} 2 & 5 & 3 \\ -3 & 4 & 1 \\ 6 & -7 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 7 \end{pmatrix}$ ______________________________
____
____

4. $\begin{pmatrix} -3 & -5 & 2 & 0 \\ 2 & 4 & 7 & -1 \\ 0 & -2 & 3 & 8 \\ -1 & 3 & 2 & -2 \end{pmatrix} \begin{pmatrix} C \\ D \\ R \\ T \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -4 \\ 7 \end{pmatrix}$ ______________________________
____
____
____

Rewrite each of the following systems of linear equations into matrix equation form:

5. $3X + 2Y - 5Z = 4$
$2X - 9Y + Z = -9$
$4X + 3Y + 8Z = 6$

6. $4R - 3T + 5V = -8$
$7R + 2T = 4$
$-5R + 3T + 9V = -1$

Kathy and David are working together on a project. On Thursday, Kathy worked for 4 hours, and David worked for 6 hours. On Friday, Kathy worked for 5 hours, and David worked for 2 hours. Together they received a total of $160.00 for their work on Thursday and $117.50 on Friday. Find the hourly rate at which each was paid. Use matrices to determine the solution to this problem. Let K = Kathy's hourly rate and D = David's hourly rate.

7. Create a system of equations to represent the given information:
Thursday's information ________________________
Friday's information ________________________
8. Rewrite this system into its matrix equation form.

The matrix showing the hours worked is called a *coefficient matrix* because its elements are the coefficients of the variables in the linear equation form. The matrix consisting of the names of the people is referred to as a *variable matrix* since it contains the variables that were used in the linear equation form. The matrix containing the "results," or the money amounts, is called a *constant matrix* since its elements are the constants in the linear equation form.

To solve the matrix equation that you have created, recall your work with linear equations:

$$5 * R = 75$$
$$1/5 * 5 * R = 1/5 * 75 \quad \text{(multiplying both sides by the multiplicative inverse of the coefficient of the variable)}$$
$$R = 15$$

Applying this same process to the matrix equation:

$$[\,C\,] * [\,X\,] = [\,P\,]$$
$$[\,C\,]^{-1} * [\,C\,] * [\,X\,] = [\,C\,]^{-1} * [\,P\,]$$
$$[\,X\,] = [\,C\,]^{-1} * [\,P\,]$$

This $[\,C\,]^{-1}$ is called the *inverse matrix* of matrix *C*. The graphing calculator or computer can generate this inverse matrix, if it exists. If using the TI–83, enter the coefficient matrix, and then call up the inverse matrix through the name menu. Use the x^{-1} key to create the inverse matrix, or use it in the solution process.

9. Using this technique, solve the matrix equation you created in #8. Show the result given on the calculator.

10. Explain the meaning of the solution given on the calculator.

Using Matrices to Solve Systems of Equations — Part II

From this part of the handout, each member of the group must choose a two *different* problems to solve — one from problems 1–5 *and* one from problems 6–10. On your own paper, show all work necessary to complete all of the steps for both problems, including the meaning of the variables, the system of linear equations, the matrix equation, the solution, and a complete sentence interpreting the result (answering the question). You may consult with the other members of your group, but ***each member must complete his/her own problems to turn in at the end of the activity.***

1. A shipping clerk finds that 3 large cartons and 2 small cartons will hold 108 lb. He also finds that 2 large cartons and 3 small cartons will hold 102 lb. How much does each size carton hold?
2. A store had a sale on two different models of calculators. One model sold for $89.75, and another model sold for $72.45. In all, 51 calculators were sold, and the total amount of sales (not including the tax) was $4,335.05. How many of each model were sold during this sale?
3. The perimeter of a rectangle is 88 cm. If the length is doubled, the perimeter is 137 cm. What are the length and width of this rectangle?
4. Jason invested $5,000, part at a yearly rate of 7% and part at a yearly rate of 9%. The total interest earned for the year was $368. How much did he invest at each rate?
5. How many liters of a 60% acid solution must be added to 8 liters of a 40% acid solution to make a mixture that is 50% acid?

---

6. Adam has 81 coins, each of which is either a penny, nickel, or dime. There are twice as many dimes as pennies. If the total value of the coins is $5.13, how many of each kind are there?
7. The greens fee for a member of a golf club is $6. The fee for a nonmember is $14, or $8 if the nonmember is a senior citizen. One day the total receipts for 226 players was $1,688. Of the nonmembers that played, the senior citizens outnumbered the others by 16. How many players were there in each category?
8. A storeowner wants to have 40 pounds of mixed nuts to sell at $4 per pound. The mixture is to contain peanuts that sell for $2.40 per pound, pecans that sell for $5.20 a pound, and cashews that sell for $6.80 per pound. If the mixture is to contain 15 more pounds of peanuts than of cashews, how many pounds of each type of nut must be used?
9. Yancy, Carl, and Mari work in a toy factory, using machines that assemble plastic boats. On Monday, Yancy and Carl operate their machines and produce 10,200 boats. On Tuesday, Carl and Mari work, producing 10,100 boats. Then on Wednesday, Yancy and Mari operate their machines to create 9,900 boats. The rates at which the machines operate is constant. How many boats can each employee produce alone in a day?
10. An oil refinery in Oklahoma sells 50% of its total production to a Chicago distributor, 20% to a Dallas distributor, and 30% to a distributor in Atlanta. Another refinery in Louisiana sells 40% of its production to Chicago, 40% to Dallas, and 20% to Atlanta. A third refinery in Texas sells the same distributors 30%, 40%, and 30% of its production. The three distributors received 219,000, 192,000, and 144,000 gallons respectively. How many gallons of oil were produced at each of the three production plants?

What is Linear Programming?

Organizing topic Systems of Linear Equations and Inequalities

Overview Students investigate linear programming techniques by using an Internet resource.

Related Standard of Learning AII.13

Objectives

- The student will solve a system of linear inequalities by graphing.
- The student will identify examples of the properties of inequality and order that occur while solving inequalities.
- The student will find the maximum and minimum values of a function over a region (linear programming).
- The student will identify the constraints in a practical situation and model them as inequalities.
- The student will graph the system of inequalities and identify the area of intersection as the feasible region. (The maximum and minimum values of the function occur at the vertices of the feasible region.)
- The student will substitute the coordinates of each vertex of the feasible region into the function to determine which vertex yields the maximum (or minimum) value of the function.
- The student will describe the results of a linear programming problem orally and in writing.

Materials needed

- Computer lab or computer with projection device suitable for viewing by all students
- Graphing calculators

Instructional activity

- In the computer lab, have students go to the Math Forum at Drexel University http://mathforum.org/library/drmath/view/52852.html and read and note the article, "Linear Programming."

Linear Programming

Organizing topic Systems of Equations and Inequalities

Overview This group activity creates a context through which students can develop their understanding of the linear programming process.

Related Standard of Learning AII.13

Objectives

- The student will solve a system of linear inequalities by graphing.
- The student will identify examples of the properties of inequality and order that occur while solving inequalities.
- The student will find the maximum and minimum values of a function over a region (linear programming).
- The student will identify the constraints in a practical situation and model them as inequalities.
- The student will graph the system of inequalities and identify the area of intersection as the feasible region. (The maximum and minimum values of the function occur at the vertices of the feasible region.)
- The student will substitute the coordinates of each vertex of the feasible region into the function to determine which vertex yields the maximum (or minimum) value of the function.
- The student will describe the results of a linear programming problem orally and in writing.

Materials needed

- Graph paper
- Computer lab or computer with projection device suitable for viewing by all students
- Graphing calculators
- A "Linear Programming Activity" handout for each student
- A "Linear Programming Homework" handout for each student

Instructional activity

1. Divide the students into groups of no more than three.
2. Have the students work together to complete the activity handout
3. Discuss the activity problem as a whole class before assigning the homework handout.

Linear Programming Activity

Many real-life problems in business and manufacturing are concerned with maximizing or minimizing a quantity — maximizing a profit, minimizing a cost of manufacturing, etc. Such situations involve a process called *optimization* and can be modeled using *linear programming*. Here the word *programming* does not refer to a computer, but to a course of action or an action plan or program. Linear programming problems tend to be somewhat lengthy, involved, multi-step problems, so it is usually important to be organized in your approach to these problems and in the recording of a solution process.

Within your group, discuss and then individually complete the problem below, which has been systematically set up for you. Make sure that you understand each step of the process. When we discuss this problem as a class, any student may be called upon to respond and explain one of the steps in the process.

Problem

An assembler of dune buggies carries two models — the Sand Buggy and the Dune Scooter. The company can use its equipment to produce up to 60 Sand Buggies and 45 Dune Scooters each month. It takes 150 hours of labor to assemble one Sand Buggy and 200 hours of labor to assemble a Dune Scooter. Due to costs of the assembly process, the company has up to 12,000 hours of labor available each month for the entire manufacturing process. If the company makes a profit of $120 on each Sand Buggy and $180 on each Dune Scooter, find the number of each model that the firm should assemble to gain the maximum profit each month.

Let S be the number of Sand Buggies and D be the number of Dune Scooters that are assembled each month by the manufacturer.

Go to the statement of the problem and underline what it says about profit. Write an equation to express the monthly profit in terms of the two models:

$P =$ ____________________

The values S and D are defined by a system of inequalities, called *constraints*. These constraints can be expressed as:

$S \geq 0$ } Explain why these two inequalities must be true.
$D \geq 0$

$S \leq ?$ ______ } Explain where these numbers come from.
$D \leq ?$ ______

Write an inequality describing the number of hours of labor that are available. _______________

On a single coordinate system, sketch the system of constraints, labeling with the appropriate variables the axes and the lines drawn. Be sure to show the shading that represents the intersection of all the half-planes. This final shaded region is called the *feasibility region* for this process. Why? The points of intersection of all of the boundary lines are called *corner points* of the feasibility region. Why?

Remember: To create the graph for each boundary line, you may choose to use the parent graph information, point-slope form, or information from the equation about the x and y intercepts.

Give the coordinates for each of the five corner points for this feasibility region. Remembering that each of these is the intersection point of two lines, calculate these instead of relying on the graph.

Linear Programming Theorem: The feasible region of a linear programming problem is convex, and the maximum or minimum quantity is determined at one of the vertices (corner points) of this region.

Why would this be true?

Take each corner point, and use it to evaluate the profit quantity given in the problem:

Corner point	Profit
________________	________________
________________	________________
________________	________________
________________	________________
________________	________________

What is the maximum profit found at the corners? _________________

At what point (coordinates) does this maximum occur? _________________

Therefore, to maximize the monthly profit, how many of each dune buggy should the firm assemble? Give the conclusion in a complete sentence.

____

____

Linear Programming Homework

Choose one of the following problems to solve with your group, and complete one other as homework. Complete each of your choices following these guidelines:

- Define the variables.
- Give the equation for the quantity to be maximized or minimized.
- Set up a system of linear inequalities for the constraints.
- Graph the feasibility region.
- Give the coordinates of all corner points.
- Evaluate the quantity at each corner point.
- Determine the solution, and state your results in a complete sentence.

1. A biologist is developing two new strains of bacteria. Each sample of Type I bacteria produces 4 new viable bacteria, and each sample of Type II produces 3 new viable bacteria. Altogether at least 240 new viable bacteria must be produced. At least 30, but not more than 60, of the original samples must be Type I. Not more than 70 of the samples are to be Type II. A sample of Type I costs $5, and a sample of Type II costs $7. If both types must be used, how many samples of each should be used to minimize the cost?

2. The Short Piano Company makes two types of pianos — the spinet and the console. The equipment in the factory allows for making at most 450 spinets and 200 consoles each month. The chart shows the costs and profit involved in making each type. During the month of June, the company can spend $360,000 in producing these pianos. To make the greatest profit, how many of each type of piano should be made in June?

	cost per unit	**profit per unit**
Spinet	$600	$125
Console	$900	$200

3. Ms. Edwards wants to invest up to $40,000 in corporate or municipal bonds, or both. She does not want to invest less than $6,000 or more than $22,000 in corporate bonds. She also has decided not to invest more than $30,000 in municipal bonds. Corporate bonds earn 8% simple interest, and municipal bonds earn 7.5%. How much should Ms. Edwards invest in each to maximize income from the interest? What is this maximum interest?

Linear Programming Problem — Maximizing Tents Produced

Organizing topic	Systems of Equations and Inequalities
Overview	This activity reinforces concepts associated with linear programming and gives students an opportunity to practice.
Related Standard of Learning	AII.13

Objectives

- The student will find the maximum and minimum values of a function over a region (linear programming).
- The student will identify the constraints in a practical situation and model them as inequalities.
- The student will graph the system of inequalities and identify the area of intersection as the feasible region. (The maximum and minimum values of the function occur at the vertices of the feasible region.)
- The student will substitute the coordinates of each vertex of the feasible region into the function to determine which vertex yields the maximum (or minimum) value of the function.
- The student will describe the results of a linear programming problem orally and in writing.

Materials needed

- Computer lab or computer with projection device suitable for viewing by all students
- Graphing calculators
- A "Linear Programming Problem" handout for each student

Instructional activity

1. Have students work alone or in partner-pairs.
2. In the computer lab, have students go to the Math Forum at Drexel University http://mathforum.org/library/drmath/view/55468.html and read and note the article, "Linear Programming Problem — Maximizing Tents Produced."
3. Distribute the handout, and allow students to work.

Linear Programming Problem — Maximizing Tents Produced

For each of the following,

a. write the relevant equation and constraining inequalities
b. graph and sketch the feasibility region
c. give the solution to the problem.

1. The Buds-R-Us Florist has to order roses and carnations for Valentine's Day. Roses cost the florist $20 per dozen and carnations cost $5 per dozen. The profit on roses is $20 per dozen and on carnations is $8 per dozen. The florist can order no more than 60 dozen flowers. How many of each kind should the florist order to maximize the profit?

   Objective equation:

   Constraints:

   Solution:

2. U. N. Scrupulous, a noted businessman in town, uses both newspaper and radio advertising each month. It is estimated that each newspaper ad reaches 8,000 people and that each radio ad reaches 15,000 people. Each newspaper ad costs $50, and each radio ad cost $100. The business can spend no more than $1,000 per month for advertising. The newspaper requires at least 4 ads be purchased each month and the radio requires that at least 5 ads be purchased each month. What combination of newspaper and radio ads should old "Scrupe" purchase in order to reach the maximum number of people?

   Objective equation:

   Constraints:

   Solution:

3. The campus store sells stadium cushions and caps. The cushions cost $1.90; the caps cost $2.25, and they sell the cushions for $5.00 and the caps for $6.00. They can obtain no more than 100 cushions and 75 caps per week. To meet demands, they have to sell a total of at least 120 of the two items together. They cannot package more than 150 per week. How many of each should they sell to maximize profit?

   Objective equation:

   Constraints:

   Solution:

Traveling Trains and Nonlinear Systems

Organizing topic Systems of Equations and Inequalities

Overview Students investigate linear programming techniques by using an Internet resource.

Related Standard of Learning AII.14

Objectives

- The student will solve linear-quadratic systems of equations algebraically and identify the set of ordered pairs that is the solution to the system
- The student will solve linear-quadratic systems of equations graphically and identify the set of ordered pairs that is the solution to the system.

Materials needed

- Computer lab or computer with projection device suitable for viewing by all students

Instructional activity

- In the computer lab, have students go to the Math Forum at Drexel University http://mathforum.org/library/drmath/view/53135.html and read and note the article, "Traveling Trains and Nonlinear Systems."

Nonlinear Systems of Equations

Organizing topic Systems of Equations and Inequalities

Overview This activity provides students with problems through which to develop and improve their understanding of nonlinear systems.

Related Standard of Learning AII.14

Objective

- The student will solve quadratic-quadratic nonlinear systems of equations algebraically and graphically and identify the solution of the system.

Materials needed

- Graphing calculators
- A "Nonlinear Systems of Equations" handout for each student
- A "Quadratic-Quadratic Systems of Equations" handout for each student

Instructional activity

- Distribute the handouts, and have the students solve the problems.

Sample resources

Mathematics SOL Curriculum Framework
 http://www.pen.k12.va.us/VDOE/Instruction/Math/math_framework.html
SOL Test Blueprints
Released SOL Test Items http://www.pen.k12.va.us/VDOE/Assessment/Release2003/index.html
Virginia Algebra Resource Center http://curry.edschool.virginia.edu/k12/algebra
NASA http://spacelink.nasa.gov/.index.html
The Math Forum http://forum.swarthmore.edu/
4teachers http://www.4teachers.org
Appalachia Educational Laboratory (AEL) http://www.ael.org/pnp/index.htm
Eisenhower National Clearinghouse http://www.enc.org/

Nonlinear Systems of Equations

1. Given this system $\begin{cases} x^2 + y^2 = 25 \\ y = x^2 - 13 \end{cases}$, complete the following:

   a. Sketch both equations on the same coordinate system.

   b. Describe both graphically and algebraically what is meant by a solution to a system of equations.

   c. How many solutions does this system have?

   d. Solve this system algebraically, using the process of substitution.

   e. Graph this system on the graphing calculator. State the window you used to “capture” this graph and the solutions on the calculator. Describe the procedure you used.

2. If possible, rewrite the system given in #1, keeping one of the equations the same but changing the other equation to satisfy each of the following stated requirements. If the change is not possible, explain why.

   a. There are exactly three solutions for the system.

   b. There are exactly two solutions for the system.

   c. There is only one solution for the system.

   d. There is no solution for the system.

3. Given this system $\begin{cases} \dfrac{x^2}{16} + \dfrac{y^2}{9} = 1 \\ x^2 - y^2 = 4 \end{cases}$, complete the following:

   a. Produce a quick sketch by hand to predict the number of solutions for this system.

   b. Solve this system, using substitution.

   c. Graph the system on the graphing calculator, and determine the solutions for this system.

4. Given a system of nonlinear equations that will graph into an ellipse, and a parabola with a horizontal axis of symmetry that will have the number of solutions specified below, state the actual solutions. If the situation is not possible, explain why.

   a. 4 solutions

   b. 2 solutions

Quadratic-Quadratic Systems of Equations

1. An ellipse intersects a circle at these points: ($\sqrt{5}$, 2), ($\sqrt{5}$, –2), (–$\sqrt{5}$, 2), and (–$\sqrt{5}$, –2). Give an example of the described system of equations.

2. A system of equations including a line and a hyperbola has 2 solutions of (–3, 0) and (5, $\frac{16}{3}$). Give an example of the described system of equations.

   Could this system have any additional solutions? If so, how many? If not, why?

3. A lab that tracks earthquakes determines that the center of an earthquake is 30 miles away. To sketch this situation, begin by graphing the location of this lab at the origin of a coordinate system, and sketch where the quake could be located. A second lab 40 miles east and 10 miles south of the first lab finds the center of the quake to be 20 miles away from its location. Sketch the location of this second lab and what its readings tell you.

   Where is the center of the earthquake? Thoroughly justify your answer.

Sample assessment

$[1 \quad 2 \quad 3 \quad 6] = P$

Matrix P shows the point value for the different ways points may be scored in a football game.

$$\begin{bmatrix} 2 \\ 0 \\ 2 \\ 3 \end{bmatrix} = S$$

Matrix S shows the number of times a team scored points in a game categorized by the way points may be scored. What was the total number of points the team scored in the game?

F 19

G 24

H 26

J 27

$$S = \begin{bmatrix} 3 \\ ^-1 \end{bmatrix}$$

$$T = [2 \quad ^-2]$$

Which matrix is the product $S \times T$?

A $[8]$

B $[6 \quad 2]$

C $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$

D $\begin{bmatrix} 6 & ^-6 \\ ^-2 & 2 \end{bmatrix}$

If $A = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$ and the product $A \cdot B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $B =$

F $\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$

G $\begin{bmatrix} ^-3 & 2 \\ 5 & ^-3 \end{bmatrix}$

H $\begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{5} & \frac{1}{3} \end{bmatrix}$

J $\begin{bmatrix} 3 & ^-4 \\ 5 & ^-8 \end{bmatrix}$

A small plant manufactures toy cars and trucks on two production lines. Matrix A is the input-output matrix of each item on each line per hour. Matrix B gives the number of hours each line operates in a day.

$$A = \begin{array}{c} \textit{Line 1} \quad \textit{Line 2} \\ \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \end{array} \begin{array}{l} \textit{Trucks} \\ \textit{Cars} \end{array} \quad \text{and}$$

Number of Hours

$$B = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$$

Which product represents the matrix of the number of toy cars and trucks produced in a day on both production lines?

A $\begin{array}{r} \textit{Trucks} \\ \textit{Cars} \end{array} \begin{bmatrix} 64 \\ 44 \end{bmatrix}$

B $\begin{array}{r} \textit{Trucks} \\ \textit{Cars} \end{array} \begin{bmatrix} 24 \\ 72 \end{bmatrix}$

C $\begin{array}{r} \textit{Trucks} \\ \textit{Cars} \end{array} \begin{bmatrix} 44 \\ 64 \end{bmatrix}$

D $\begin{array}{r} \textit{Trucks} \\ \textit{Cars} \end{array} \begin{bmatrix} 16 \\ 48 \end{bmatrix}$

$$\begin{cases} -x + 2y - 5z = -12 \\ 2x - 3y + 7z = 19 \\ -5x - 2y + z = -10 \end{cases}$$

Given the system of equations above,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = ?$$

F $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

G $\begin{bmatrix} -12 \\ 19 \\ 10 \end{bmatrix}$

H $\begin{bmatrix} 1.1 & 0.8 & -0.1 \\ -3.7 & -2.6 & -0.3 \\ -1.9 & -1.2 & -0.1 \end{bmatrix}$

J $\begin{bmatrix} 0 \\ -11 \\ 32 \end{bmatrix}$

Which matrix is the multiplicative inverse of $\begin{bmatrix} 7 & 16 \\ 4 & 9 \end{bmatrix}$?

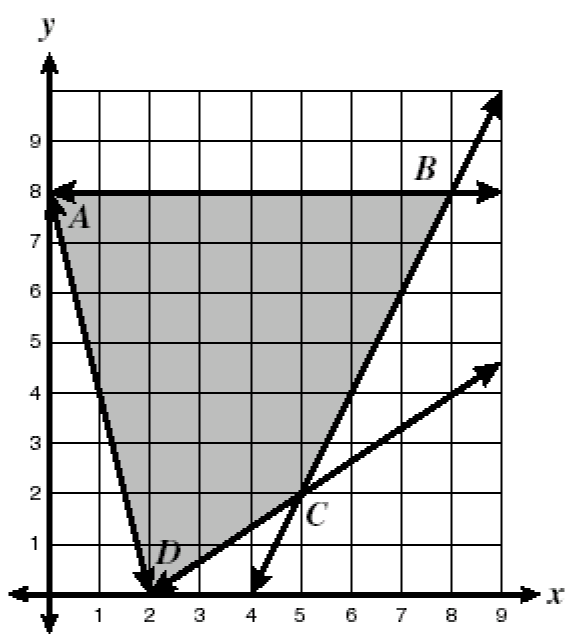
A $\begin{bmatrix} ^-9 & 4 \\ 4 & 7 \end{bmatrix}$

B $\begin{bmatrix} ^-9 & 16 \\ 4 & ^-7 \end{bmatrix}$

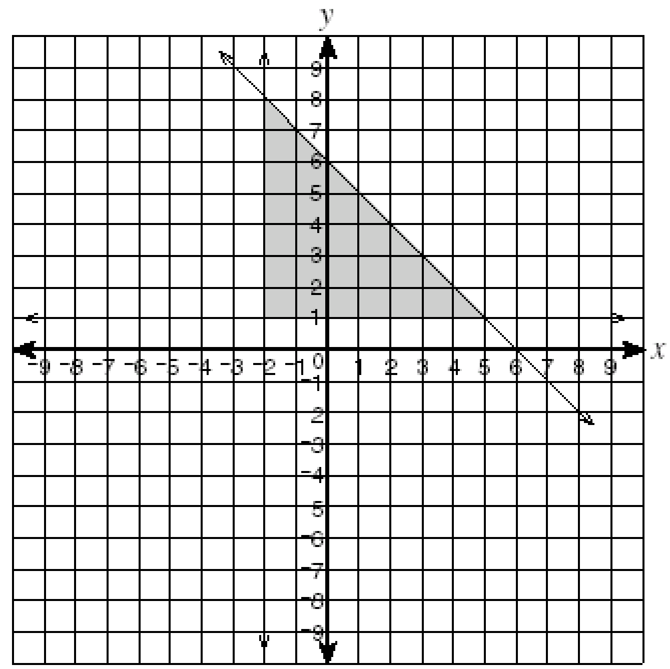
C $\begin{bmatrix} 9 & 16 \\ 4 & 7 \end{bmatrix}$

D $\begin{bmatrix} ^-9 & 16 \\ 4 & 7 \end{bmatrix}$

The graph below of the linear programming model consists of polygon *ABCD* and its interior. Under these constraints, which is the point where the *maximum* value of $4x + 3y$ occurs?



F *A*
G *B*
H *C*
J *D*



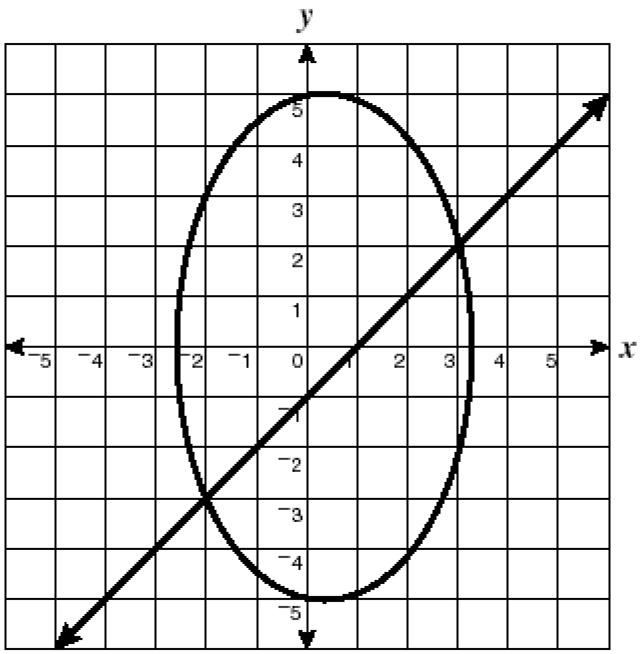
Which set of constraints produced the shaded feasible region shown above?

A $\begin{cases} y \geq 6 - x \\ x \geq 1 \\ y \geq {}^-2 \end{cases}$

B $\begin{cases} x + y \leq 6 \\ x \geq {}^-2 \\ y \geq 1 \end{cases}$

C $\begin{cases} x \geq 6 - y \\ x \geq 1 \\ y \leq 2 \end{cases}$

D $\begin{cases} x + y \leq 8 \\ x - y \leq 6 \\ x \geq {}^-2 \\ y \geq 1 \end{cases}$



This is a portion of the graph of a system of equations. Which is most likely the solution set for the system?

F {(–2.1, –3.4), (2, 3)}

G {(–3, 2), (–2.1, 3.4)}

H {(–2, –3), (3, 2)}

J {(2.1, –3.4), (3, 2)}

$$\begin{cases} y = x + 1 \\ y = (x + 3)^2 - 4 \end{cases}$$

Which is the solution set for the system of equations above?

A {(–6.4, 7.4), (–0.6, 1.6)}

B {(–4, –3), (–2, –3)}

C {(–4, –3), (–1, 0)}

D {(0.6, 1.6), (6.4, 7.4)}

Organizing Topic A Transformation Approach to Relations and Functions

Standards of Learning

AII.1 The student will identify field properties, axioms of equality and inequality, and properties of order that are valid for the set of real numbers and its subsets, complex numbers, and matrices.

AII.4 The student will solve absolute value equations and inequalities graphically and algebraically. Graphing calculators will be used as a primary method of solution and to verify algebraic solutions.

AII.5 The student will identify and factor completely polynomials representing the difference of squares, perfect square trinomials, the sum and difference of cubes, and general trinomials.

AII.6 The student will select, justify, and apply a technique to solve a quadratic equation over the set of complex numbers. Graphing calculators will be used for solving and for confirming the algebraic solutions.

AII.8 The student will recognize multiple representations of functions (linear, quadratic, absolute value, step, and exponential functions) and convert between a graph, a table, and symbolic form. A transformational approach to graphing will be employed through the use of graphing calculators.

AII.9 The student will find the domain, range, zeros, and inverse of a function; the value of a function for a given element in its domain; and the composition of multiple functions. Functions will include exponential, logarithmic, and those that have domains and ranges that are limited and/or discontinuous. The graphing calculator will be used as a tool to assist in investigation of functions.

AII.10 The student will investigate and describe through the use of graphs the relationships between the solution of an equation, zero of a function, *x*-intercept of a graph, and factors of a polynomial expression.

AII.15 The student will recognize the general shape of polynomial, exponential, and logarithmic functions. The graphing calculator will be used as a tool to investigate the shape and behavior of these functions.

AII.18 The student will identify conic sections (circle, ellipse, parabola, and hyperbola) from his/her equations. Given the equations in *(h, k)* form, the student will sketch graphs of conic sections, using transformations.

AII.19 The student will collect and analyze data to make predictions and solve practical problems. Graphing calculators will be used to investigate scatterplots and to determine the equation for a curve of best fit. Models will include linear, quadratic, exponential, and logarithmic functions.

Essential understandings, knowledge, and skills	Correlation to textbooks and other instructional materials
• Recognize the graphs and equations of parent functions such as $y = x$, $y = x^2$, $y = x^3$, $y = \|x\|$, $y = a^x$, step functions, and other polynomial functions.	____________________
• Apply transformations (translations, reflections, dilations, and rotations) and combinations of transformations to parent graphs.	____________________

- Given an image graph of a function, describe the transformations that were performed on the pre-image and the order in which they could have occurred. ____________________________
- Given the equation of a parent graph, vary the coefficients and constants of the equation, observe the changes in the graph of the parent, and generalize the changes to the graphs of other functions. ____________________________
- Build a strong connection between the algebraic and geometric representations of functions. ____________________________
- Given an equation of a function, sketch the graph of the function. ____________________________
- Collect data, and display it in a scattergraph. ____________________________
- Analyze data, using measures of central tendency (mean, median, and mode), the range of the data, and box-and-whisker plots. ____________________________
- Determine the equation of the curve of best fit, using the graphing calculator. Consider the graphs of the parent functions when determining which curve might be appropriate. Use the equation to make predictions. ____________________________
- Investigate the commutativity and associativity of combinations of transformations. ____________________________
- Given the graph or the equation of a function, identify the domain and range of the function. Include functions with discontinuities. ____________________________
- Find the value of a function for a given element in the domain. ____________________________
- Find the composition of functions algebraically and graphically. ____________________________
- Find the inverse of a function algebraically and graphically. ____________________________
- Explain how composition of functions and finding the inverse of a function affect the domain and range of the functions. ____________________________
- Demonstrate that the exponential and logarithmic functions are inverse functions. ____________________________
- Factor polynomials completely (difference of two squares, perfect square trinomials, general trinomials, sum and difference of cubes). ____________________________
- Find the zeros of a function algebraically and graphically. Quadratic equations may be solved using a variety of techniques that include but are not limited to factoring to use the zero product property, square roots, the quadratic formula, completing the square, and graphing. ____________________________

- Investigate the relationship between the solutions of an equation, zeros of a function, *x*-intercepts, and factors of a polynomial. __________________________
- Identify examples of the field properties of real numbers and the properties of equality that occur while solving equations. __________________________
- Given the zeros of a polynomial, write an equation for the polynomial function. __________________________
- Define *absolute value*. __________________________
- Evaluate expressions that contain absolute value. __________________________
- Recognize that the value of the argument for absolute value must be greater than or equal to zero, and explain why. __________________________
- Solve absolute value equations algebraically and graphically. __________________________
- Identify examples of the field properties of real numbers and the properties of equality that occur while solving equations. __________________________
- Solve absolute value inequalities algebraically and graphically. __________________________
- Identify examples of the properties of inequality and order that occur while solving inequalities. __________________________
- Recognize the graphs of the conic sections defined as any figure that can be formed by slicing a double cone (parabola, ellipse, circle, and hyperbola). __________________________
- Given an equation, identify the conic section, and graph it transformationally. __________________________

Families of Functions

Organizing topic A Transformational Approach to Relations and Functions

Overview Students explore families of functions, using transformations as the common theme.

Related Standards of Learning AII.1, AII.8, AII.15

Objectives

- The student will recognize the graphs and equations of parent functions such as $y = x$, $y = x^2$, $y = x^3$, $y = |x|$, $y = a^x$, step functions, and other polynomial functions.
- The student will apply transformations (translations, reflections, dilations, and rotations) and combinations of transformations to parent graphs.
- The student will, given an image graph of a function, describe the transformations that were performed on the pre-image and the order in which they could have occurred.
- The student will, given the equation of a parent graph, vary the coefficients and constants of the equation, observe the changes in the graph of the parent, and generalize the changes to the graphs of other functions.
- The student will build a strong connection between the algebraic and geometric representations of functions.
- The student will, given an equation of a function, sketch the graph of the function.
- The student will investigate the commutativity and associativity of combinations of transformations.

Materials needed

- Graphing calculators
- Patty paper
- Graph paper
- A “Coordinates and Symmetry” handout for each student
- A “Matching Quadratic Equations, Graphs, and Tables” handout for each student

Instructional activity I: Review of transformations from Grade 8 mathematics

Translations

1. Have students draw a simple shape on the patty paper and put a dot in the interior of the shape.
2. Next, have students draw a ray from the dot to an edge of the paper and mark a second dot on the ray outside the simple shape. Students will translate the shape along this ray, according to the following steps. This is a good time to discuss the meaning of the word *translate* as it is used in geometry, as opposed to its common meaning with languages.
3. On a second sheet of patty paper, ask the students to mark the distance between the two dots. This is the translation distance.
4. Have students place a third sheet of patty paper over the first and trace the shape, ray, and interior dot. Now ask students to carefully slide the interior dot on the tracing along the ray until it is on top of the second dot on the first paper.
5. Tell the students to use the patty paper with the translation distance marked on it to measure the distance between corresponding points on the two figures. Encourage students to discuss their

findings. Lead the discussion toward the fact that a translated figure is congruent to the original except that its location in the plane is different. Make certain that students note that each point moves by the same distance and in the same direction during translation.

Rotations

6. Ask students to draw a simple shape on patty paper and place a dot on an edge of the shape or in the interior of the shape and a second dot outside the shape. The second dot will be the center of rotation.
7. Next, tell students to draw a ray from the second dot through the first and draw a second ray from the center of rotation (second dot) to create an acute angle of rotation.
8. Have students place a second patty paper over the first and trace the figure, the dots, and the first ray. Using a pencil to hold the two sheets of patty paper together at the center of rotation, turn the second patty paper until the tracing of the first ray is aligned with the second ray.
9. Ask students to place a third sheet of patty paper over this rotation and carefully trace the two figures and the angle of rotation. They will use this tracing for the remainder of the exercise.
10. Allow students to use another patty paper or a ruler to measure the distance between corresponding parts of the two figures. What patterns do they notice? How is this finding different from a translation transformation?
11. Have them use another patty paper and a straightedge to create an angle with the center of rotation as its vertex, a point on the original figure along one ray and the corresponding point on the rotated figure along its other ray. What do they notice about this angle and the original angle of rotation? Does this finding hold true for all parts of the rotated figure?

Reflections

12. Ask students to draw a simple shape on patty paper and then fold the paper so that the fold does not intersect the shape. This fold crease is the line of reflection.
13. Have students fold the paper along the line of reflection and trace the figure onto the other side of the paper.
14. Have students draw two segments, each connecting a point on the original figure with its corresponding point on the reflected figure. Use patty paper to measure the lengths of these segments. Encourage students to discuss their findings and describe what happens when a figure is reflected.
15. To extend this activity, encourage students to think about the relationship between the line of reflection and the segments connecting corresponding parts of the two figures. They should notice that the line of reflection is perpendicular to the segments and that the line bisects each segment.
16. Ask students to mark the x- and y-axes on their graph paper. They can construct two sets of axes ranging from –10 to +10 on one sheet of graph paper.
17. On the first set of axes, have students mark the x-axis as the line of symmetry. On the second set of axes, have them mark the y-axis as the line of symmetry. On “Coordinates and Symmetry” handout, have them record the coordinates of each point as it is reflected across the line of symmetry indicated. Have the students then generalize the pattern for any coordinates (x, y).

Follow-up/extension

- Use the activity formats on the handout to devise similar activities for dilations and to provide opportunities for students to explore the effect of other transformations on the coordinates of points in the plane.

Instructional activity II: Review of parent equations and families of equations from Algebra I

1. Have students build a table of values for $y = x$ and graph the ordered pairs. Have them then verify their graph by entering the equation into the graphing calculator and using ZOOM DECIMAL and TRACE to verify that points are on the line.
2. Have students work in small groups to design an exploration of the graph of $y = a(x + b) + c$. If you think the students need guidance, try the following sequence of steps.
    a. Enter Y1 = x in the graphing calculator. This is the graph of the parent function. If the equation of the parent is $y = ax$, what is the value of a in $y = x$?
    b. Graph several examples of $y = ax$ by using a variety of values for a. For example, enter Y2 = $1.3x$; Y3 = $2x$; Y4 = $2.3x$. In each of these examples, $a > 1$. What effect does a have on the graph of the function when $a > 1$?
    c. Graph several examples of $y = ax$ by using a variety of values for a when $a < 1$ but is still positive. What effect does a have on the graph of the function when $0 < a < 1$?
    d. What is the effect on the graph of the function $y = ax$ when $a = 0$? Explain why the graph looks the way it does.
    e. Summarize the effect a has on the graph of $y = ax$ in terms of transformations.
    f. Let a assume some negative values. What happens to the graph of the function? Describe the effect in terms of transformations.
    g. Let's explore the effect of c on the graph of $y = x + c$. Use a variety of values for c. Record your work. Describe the effect in terms of transformations.
    h. Try to predict what the graph of the equations below will look like before you try them on the graphing calculator. Sketch your predicted graph. Write an explanation for your prediction.
3. Have small groups of students design an exploration of the graph of $y = a(x + b)^2 + c$. Students may use the same sequence of events described for exploring $y = ax + b$. The teacher may start the experiment by having students build a table of values for $y = x^2$ and use the ordered pairs generated to graph the parent. Have the students verify the graph, using the graphing calculator.
4. Another parent to be graphed is $y = x^3$. Encourage students to build a table of values and generate a graph of the parent. Have them verify the graph, using the graphing calculator.
5. For the graph of $y = a(x + b)^3 + c$, have students describe the effect of a, b, and c on the parent graph in terms of transformations.
6. Have students continue to explore parent functions, such as $y = |x|$, $y = ax$, $y = \log x$, and $y = [x]$. Students should be able to predict graph behavior in terms of transformations. Give students graphs of functions, and allow them to experiment with finding the equation from the given graph. Predictions may be verified by using the graphing calculator.

Homework

- Journal: Have students explore the commutative property and associative property for combinations of transformations and show their work to support their conclusions.

Coordinates and Symmetry

1. Draw two sets of axes on your graph paper. Each set of axes should include the range –10 to +10.

2. On your first set of axes, mark the *x*-axis as the line of reflection.

3. Plot the points listed below on your first set of axes. Then reflect each point across the line of symmetry, and write in the coordinates of the reflection.
   (–6, 1) → (__, __)
   (2, 3) → (__, __)
   (–2, –2) → (__, __)
   (0, 4) → (__, __)
   (4, 0) → (__, __)

4. What pattern do you see in the new coordinates? In general, if a point (*x*, *y*) is reflected across the *x*-axis, the coordinates of the reflection are (__, __).

5. On the second set of axes, mark the *y*-axis as the line of reflection.

6. Plot the points listed below on your second set of axes. Then reflect each point across the line of symmetry, and write in the coordinates of the reflection.
   (-6, 1) → (__, __)
   (2, 3) → (__, __)
   (-2, -2) → (__, __)
   (0, 4) → (__, __)
   (4, 0) → (__, __)

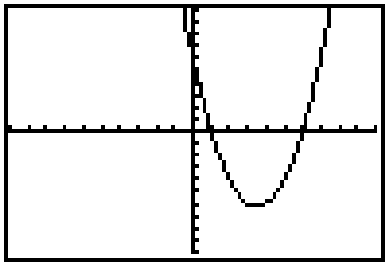
7. What pattern do you see in the new coordinates?

Matching Quadratic Equations, Graphs, and Tables

Cut out each rectangle, and glue it to an index card. Shuffle the cards. Match each quadratic equation to its graph and to its table of values.

WINDOW FORMAT
Xmin=-10
Xmax=10
Xscl=1
Ymin=-10
Ymax=10
Yscl=1

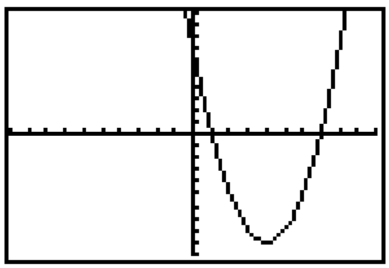
$Y_1 = X^2 - 7X + 6$
$Y_2 =$
$Y_3 =$
$Y_4 =$
$Y_5 =$
$Y_6 =$
$Y_7 =$
$Y_8 =$



X	Y1
1	0
6	0
0	6

X=

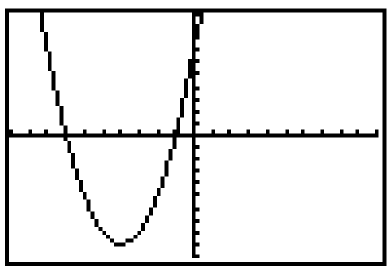
$Y_1 = X^2 - 8X + 7$
$Y_2 =$
$Y_3 =$
$Y_4 =$
$Y_5 =$
$Y_6 =$
$Y_7 =$
$Y_8 =$



X	Y1
1	0
7	0
0	7

X=

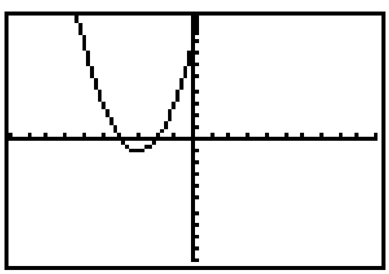
$Y_1 = X^2 + 8X + 7$
$Y_2 =$
$Y_3 =$
$Y_4 =$
$Y_5 =$
$Y_6 =$
$Y_7 =$
$Y_8 =$



X	Y1
-7	0
-1	0
0	7

X=

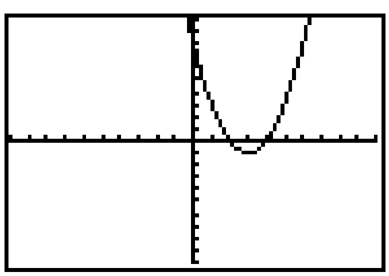
$Y_1 = X^2 + 6X + 8$
$Y_2 =$
$Y_3 =$
$Y_4 =$
$Y_5 =$
$Y_6 =$
$Y_7 =$
$Y_8 =$



X	Y1
-4	0
-2	0
0	8

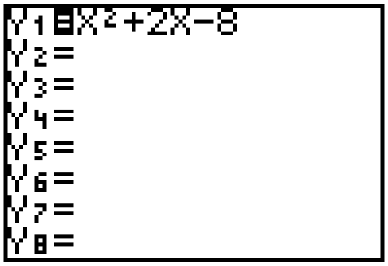
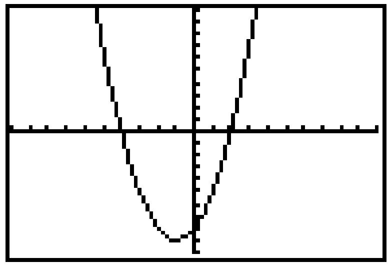
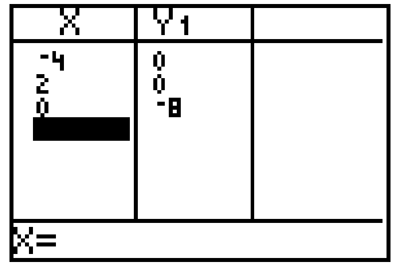
X=

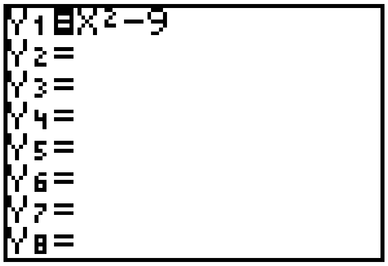
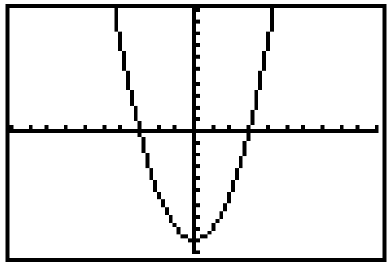
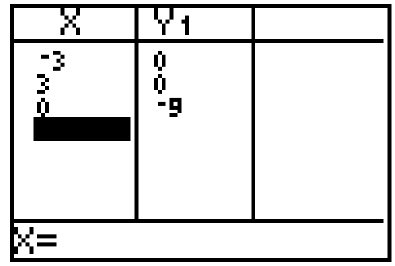
$Y_1 = X^2 - 6X + 8$
$Y_2 =$
$Y_3 =$
$Y_4 =$
$Y_5 =$
$Y_6 =$
$Y_7 =$
$Y_8 =$

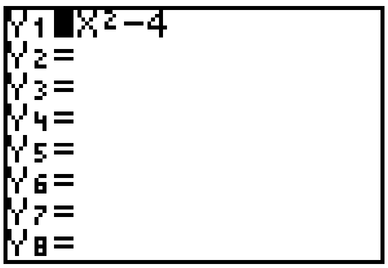
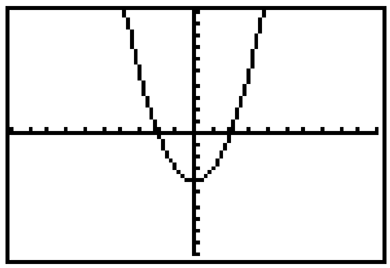
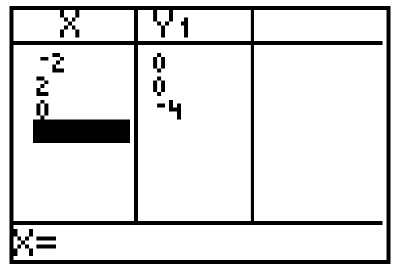


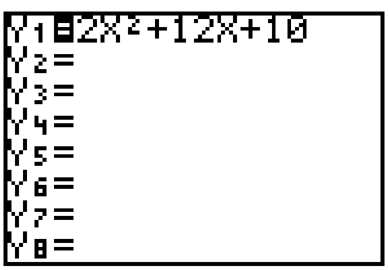
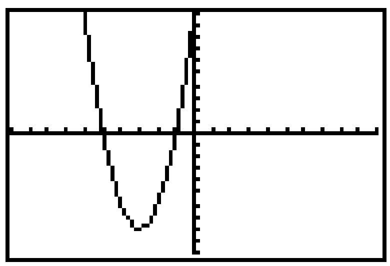
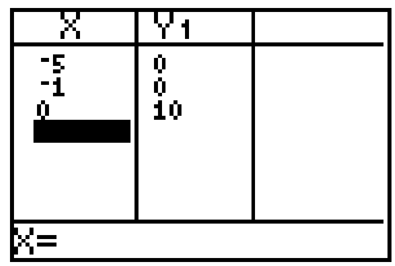
X	Y1
2	0
4	0
0	8

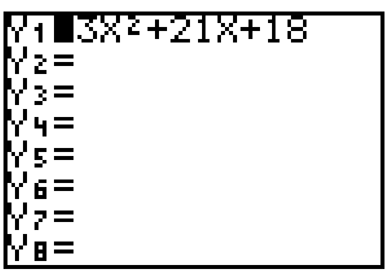
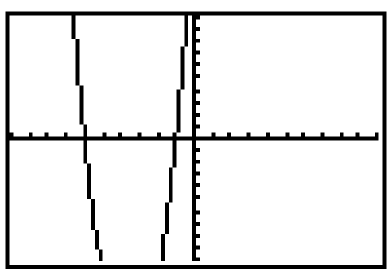
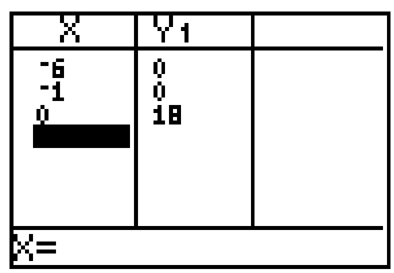
X=

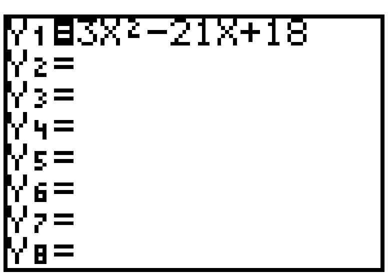
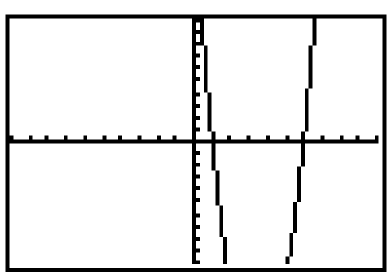
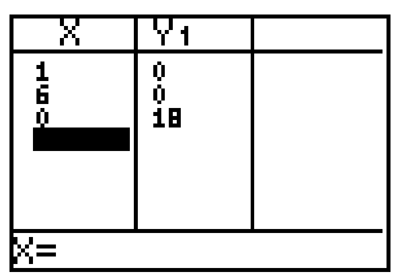

$Y_1 = X^2+2X-8$
$Y_2=$
$Y_3=$
$Y_4=$
$Y_5=$
$Y_6=$
$Y_7=$
$Y_8=$


X
Y_1
-4
2
0
0
0
-8
X=

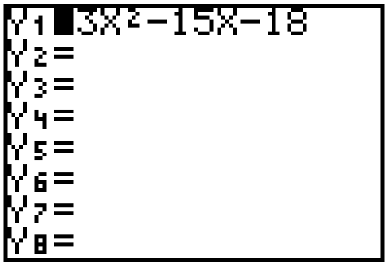

$Y_1 = X^2-9$
$Y_2=$
$Y_3=$
$Y_4=$
$Y_5=$
$Y_6=$
$Y_7=$
$Y_8=$


X
Y_1
-3
3
0
0
0
-9
X=

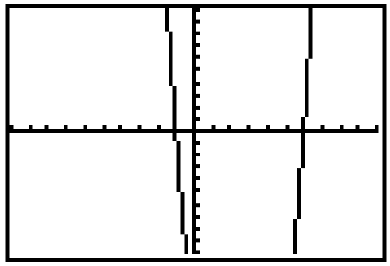

$Y_1 = X^2-4$
$Y_2=$
$Y_3=$
$Y_4=$
$Y_5=$
$Y_6=$
$Y_7=$
$Y_8=$


X
Y_1
-2
2
0
0
0
-4
X=

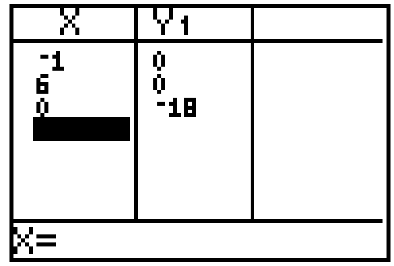

$Y_1 = 2X^2+12X+10$
$Y_2=$
$Y_3=$
$Y_4=$
$Y_5=$
$Y_6=$
$Y_7=$
$Y_8=$


X
Y_1
-5
-1
0
0
0
10
X=

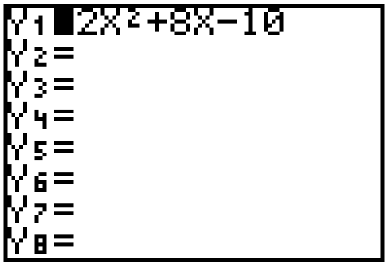

$Y_1 = 3X^2+21X+18$
$Y_2=$
$Y_3=$
$Y_4=$
$Y_5=$
$Y_6=$
$Y_7=$
$Y_8=$


X
Y_1
-6
-1
0
0
0
18
X=

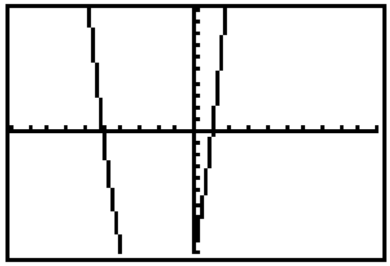

$Y_1 = 3X^2-21X+18$
$Y_2=$
$Y_3=$
$Y_4=$
$Y_5=$
$Y_6=$
$Y_7=$
$Y_8=$


X
Y_1
1
6
0
0
0
18
X=

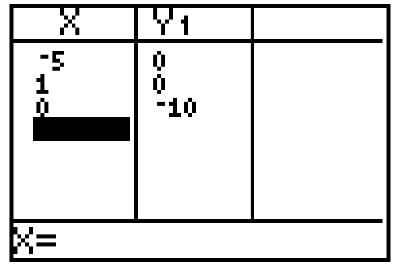

$Y_1 = 3X^2-15X-18$
$Y_2=$
$Y_3=$
$Y_4=$
$Y_5=$
$Y_6=$
$Y_7=$
$Y_8=$

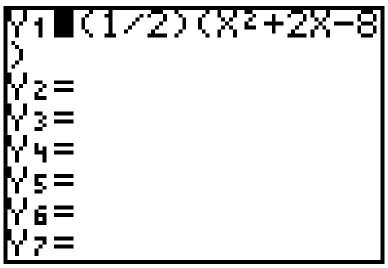


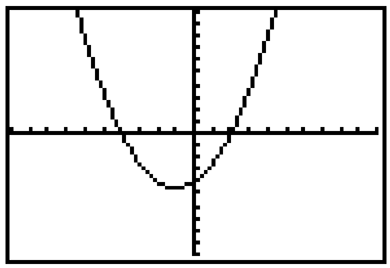

X Y1
-1 0
6 0
0 -18
X=

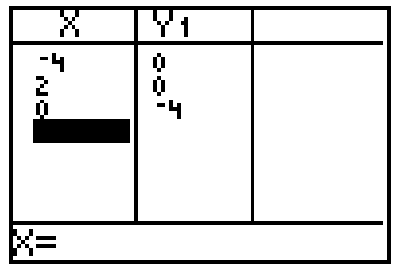

$Y_1 = 2X^2+8X-10$
$Y_2=$
$Y_3=$
$Y_4=$
$Y_5=$
$Y_6=$
$Y_7=$
$Y_8=$

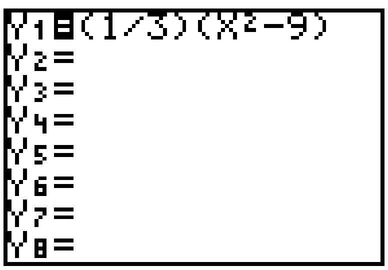


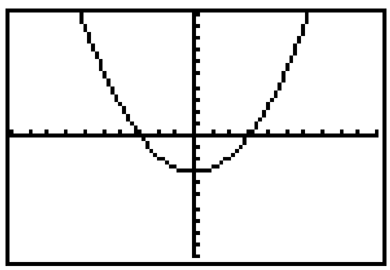

X Y1
-5 0
1 0
0 -10
X=

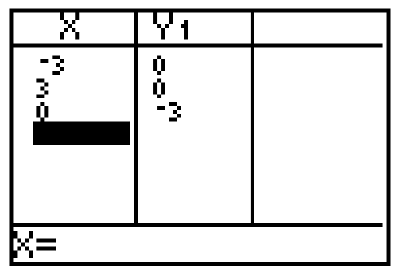

$Y_1 = (1/2)(X^2+2X-8)$
$Y_2=$
$Y_3=$
$Y_4=$
$Y_5=$
$Y_6=$
$Y_7=$

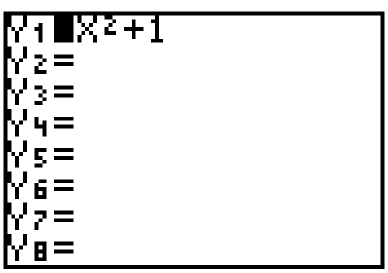


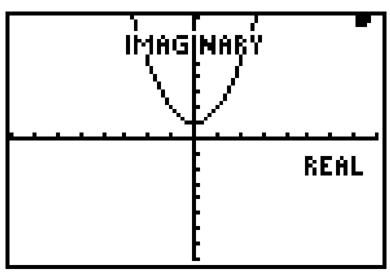

X Y1
-4 0
2 0
0 -4
X=

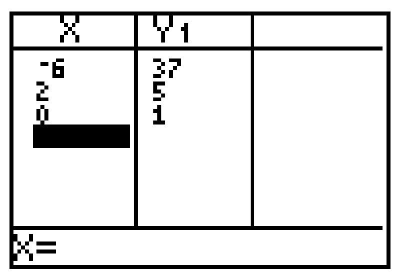

$Y_1 = (1/3)(X^2-9)$
$Y_2=$
$Y_3=$
$Y_4=$
$Y_5=$
$Y_6=$
$Y_7=$
$Y_8=$

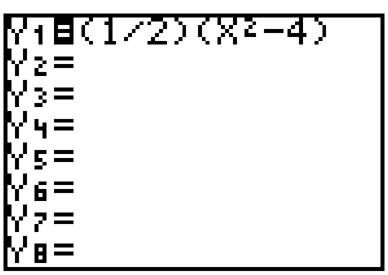


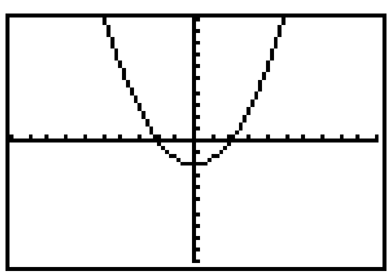

X Y1
-3 0
3 0
0 -3
X=

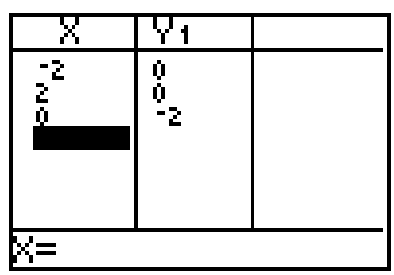

$Y_1 = X^2+1$
$Y_2=$
$Y_3=$
$Y_4=$
$Y_5=$
$Y_6=$
$Y_7=$
$Y_8=$

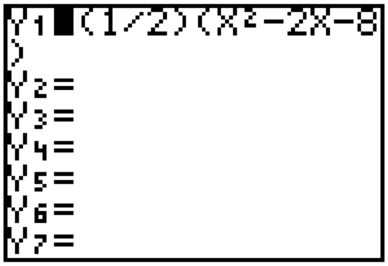

IMAGINARY
REAL

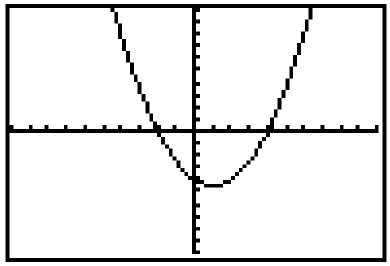

X Y1
-6 37
2 5
0 1
X=

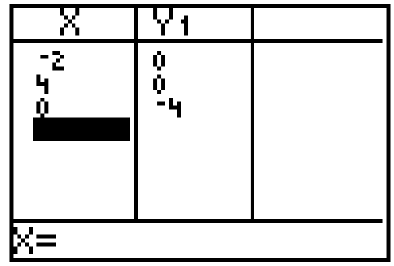

$Y_1 = (1/2)(X^2-4)$
$Y_2=$
$Y_3=$
$Y_4=$
$Y_5=$
$Y_6=$
$Y_7=$
$Y_8=$

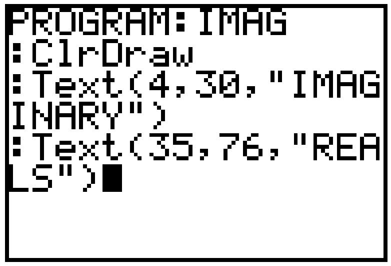



X Y1
-2 0
2 0
0 -2
X=


Y1=(1/2)(X²-2X-8
)
Y2=
Y3=
Y4=
Y5=
Y6=
Y7=




X
Y1
-2
4
0
0
0
-4
X=


PROGRAM:IMAG
:ClrDraw
:Text(4,30,"IMAG
INARY")
:Text(35,76,"REA
LS")

Collecting Data and Regression Equations

Organizing topic A Transformation Approach to Relations and Functions
Overview Students are involved in exploring data generated from a variety of sources.
Related Standards of Learning AII.15, AII.19

Objectives

- The student will collect data and display it in a scattergraph.
- The student will determine the equation of the curve of best fit, using the graphing calculator.
- The student will consider the graphs of the parent functions when determining which curve might be appropriate.
- The student will use the equation to make predictions.

Materials needed

- Circular regions with varying radii (1 cm, 2 cm, 3 cm, 4 cm, 5 cm, etc.) cut out of card stock
- One-centimeter grid paper
- Graphing calculators
- Large cup for each group
- Large bag of M&M's® for each group
- Napkins or paper towels to cover desktops
- A copy of each of the five handouts for each students

Instructional activity 1

Note: The more circular regions the students have to measure, the better the scattergraph will look. This activity may be done with younger students as well. Instead of finding the regression equation for the data, students may also find the mean area as determined by the students in the class. (Standard A.18)

1. Give each student an Activity 1 handout, "Collecting Data and Regressions."
2. Have the students work alone or in pairs.

Instructional activity 2

Note: This is a very interesting situation. During certain years, the regression "appears" to be linear, while at other times it takes on different shapes. This could be used as an introduction into linearization in calculus.

1. Give each student an Activity 2 handout, "Collecting Data and Regressions."
2. Have the students work alone or in pairs.

Instructional activity 3

1. Divide the class into groups of no more than three.
2. Distribute M&M's, cups, and Activity 3 handout, "M&M's® Experiment."
3. In the data collection process, the number of M&M's remaining usually does not go to zero due to the manufacturing process. If the number actually does go to zero, this data point needs to be recorded as 0.01

Instructional activity 4

1. Make sure that the students understand that stringed instruments, like violins and guitars, produce different pitches of a musical scale depending on the length of the string and the frequency of the vibrating string. When under equal tension, the frequency of the vibrating string varies inversely with the string length.
2. Give each students an Activity 4 handout, "Collecting Data and Regressions."
3. Have the students work alone.

Homework

- Have the students complete "The Ewok-Jawa Problem" detailed on the handout.

Activity 1: Collecting Data and Regressions

1. Place the circular regions that have been provided by the teacher on the centimeter grid paper, and trace them.
2. Count the number of square centimeters in the area of each circular region, and record the measurements in the table:

Circle radius	No. of square centimeters
1	
2	
3	
4	
5	
6	

3. Record the radius of each circle in L1 in the calculator and the area of the circular region in L2.
4. Use the calculator to generate a scattergraph.
5. Examine the scattergraph carefully, and decide into which family of functions the graph might fit.
6. Using the regression equation capabilities of the calculator and your knowledge of function families, find the regression equation that you believe best fits your data, and graph the curve through the data points.
7. Using the equation of the curve, predict the area of a circular region with a radius of 12 cm.
8. What would be the radius of a circular region that covers 450 square cm? (Hint: Work backwards to find the answer.)

Activity 2: Collecting Data and Regressions

Year	Marriages	Divorces
1960	1,523,000	393,000
1962	1,557,000	413,000
1964	1,725,000	450,000
1966	1,857,000	499,000
1968	2,069,258	584,000
1970	2,158,802	708,000
1972	2,282,154	845,000
1974	2,229,667	977,000
1976	2,154,807	1,083,000
1978	2,282,272	1,130,000
1980	2,406,708	1,182,000
1982	2,495,000	1,180,000
1984	2,487,000	1,155,000
1986	2,400,000	1,159,000
1988	2,389,000	1,183,000
1990	2,448,000	1,175,000
1992	2,362,000	1,215,000

1. Using the given data representing the number of marriages and divorces in the U.S. from 1960 to 1992, enter the year in List 1, the total number of marriages in List 2, and the total number of divorces in List 3.
2. Calculate the divorce rate, and put the data in List 4.
3. Which type of regression best fits the data from 1960 to 1976?
4. If the pattern continued in this way, what would the predicted divorce rate be in 1994?
5. Which type of regression best fits the data from 1978 to 1992?
6. Based on your data, what would the predicted divorce rate be for the year 1995?

Extension

7. What is a possible explanation for the different regressions in the marriage and divorce rates?
8. Do you find anything “odd” about this data?

Activity 3: M&M's® Experiment

The following experiment is designed to simulate a natural occurrence.

Part I. Data Collection

1. Count the M&M's® in the cup that was given to your group, and record this number in the chart to the right of toss no. 0.
2. Shake the cup, and carefully dump the M&M's® on the desktop. Remove and set aside those candies that landed with the *m* showing. Count the ones remaining, and record this number in the chart to the right of toss #1. Place these candies back in the cup.
3. Shake the cup, dump the candy, remove and set aside the ones with the *m* showing, count the remaining, and record.
4. Keep repeating the process, recording the results each time until no *m*'s appear.
5. Graph on your graph paper the collected data from the chart.

No. of the toss	No. remaining
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	

Part II. Analysis of Data

6. The data collected can be modeled with an exponential curve of the form $y = ab^x$. Using the graph from Part I, develop an equation to fit the data collected.

7. Give the coordinates of the beginning point in your data.

8. What information does this give about the equation being developed?

9. Locate a second point on the graph.

10. Using this specified point and the original data point, find the value of *b* in the general equation.

11. Write the completed equation for the graph.

12. How well does this equation predict the remaining data on the graph? Check the generated value for *y* and the graphed value for *y* for several chosen values of *x*.

13. How "good" a fit is the equation that you developed?

Part III. TI-83 Graph of the Data

14. The graphing calculator can assist in the development of the equation for the M&M's data.
15. STAT ClrList L1, L2, L3, L4, L5, L6 ENTER This command will clear any previous data.
16. STAT EDIT Enter the trial number in List 1 and the number remaining in List 2.
17. WINDOW Adjust the viewing window to adequately picture the collected data. ZOOM STAT will set an appropriate window automatically.
18. STATPLOT PLOT1 ENTER Choose the type of graph that will picture that data.
19. Y1 = Enter the equation you developed in Part II, and graph this over the data points.
20. How well does your equation "fit" the data points? Any problem areas? Any discrepancies? How accurate is your equation?

Part IV. TI-83 Analysis of the Data

21. STAT CALC A:ExpReg ENTER The calculator will evaluate the data and determine the "best fit" exponential curve in the form of $y = ab^x$ for the data.
22. Give the equation generated by the calculator for this data.

23. What is the correlation coefficient (r) for this equation?

24. What does this mean?

25. Y = Y2 VARS 5:STATISTICS EQ 1:RegEq ENTER This will enter the generated regression equation into the graphing feature of the calculator. ExpReg Y1 or ExpReg Y2 will insert the equation into Y1 or Y2.
26. Graph this equation over the original data points.
27. Use GRAPH Y = Select Y1, put the cursor over the = symbol, and press ENTER to "turn off" the equation in Y1 and view only the original data and the calculator regression equation.
28. How well does this equation match or "fit" the data? Any problem areas?

29. How does this equation compare with the one manually generated in Part II?

Extension

- Describe some natural occurrence that could be depicted with data such as that collected. Thoroughly explain why this data would, in general, fit the occurrence you are choosing.

Activity 4: Music and Mathematics

Stringed instruments, like violins and guitars, produce different pitches of a musical scale depending on the length of the string and the frequency of the vibrating string. When under equal tension, the frequency of the vibrating string varies inversely with the string length.

1. Complete the table to find the string lengths for a C major scale. Round your answers to the nearest whole number.

Pitch	C	D	E	F	G	A	B	C
Frequency (cycles/sec.)	**523**	**587**	**659**	**698**	**784**	**880**	**988**	**1046**
String length (mm)	**420**							

2. Find a function that models this variation.

3. Describe how the values of the frequency change in relation to the string length.

4. Make a scattergraph of your data. Describe the graph.

5. What is true about the product of the frequency and string length for the first pitch, C?

6. What is true about the product of the frequency and string length for the second pitch, D?

7. How can you determine the value of the string length for the second pitch, D?

The Ewok-Jawa Problem

1. A band of 45 Ewoks crash-landed in the National Forest last night. This sounds like a small problem, but the population will grow at the rapid rate of 22% per year. Write an equation to describe the population in any given year. Also create a table that shows the Ewok population every five years from this year to the year 2050.

2. Coincidentally, a band of Jawas crash-landed near the Ewoks. The Jawas have a population growth modeled by the equation $A = 105(0.91)^t$, where A is the population at any time, t, given in years. Is this population increasing? Decreasing? Stagnant? Explain your reasoning.

3. If in 2005 the two tribes begin fighting so that the casualties of war offset the Ewok birth rate, write the compound equation to describe the Ewok population from 2003 to 2050.

4. If the Jawas lose 15% of their members each year in battle, when will their tribe be threatened by extinction?

5. Graph, using the TI-83, and label the equations for each scenario #1–4. Include a brief explanation of the graph's implications.

Comparative Stats: Box-and-Whisker Plots

Organizing topic A Transformational Approach to Relations and Functions
Overview Students extend their experience with box-and-whisker plots.
Related Standard of Learning AII.19

Objective

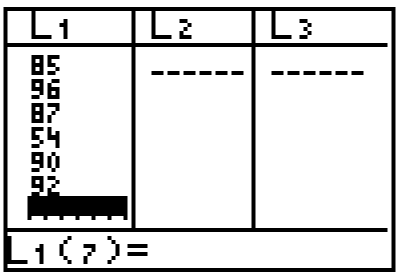
- The student will analyze data, using measures of central tendency, range, and box-and-whisker plots.

Materials needed

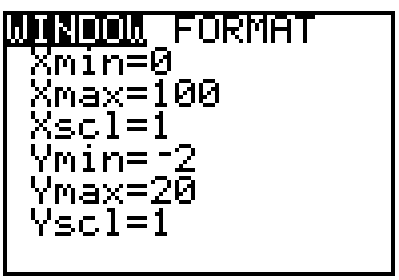
- Graphing calculators

Instructional activity 1

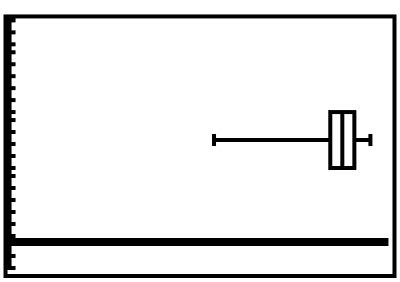
1. Manually make a box-and-whiskers plot of the scores on this Statistics exam.
   85 96 87 54 90 92
2. Now enter the scores into L_1.



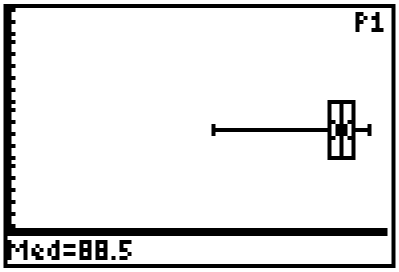
3. Choose an appropriate Window.

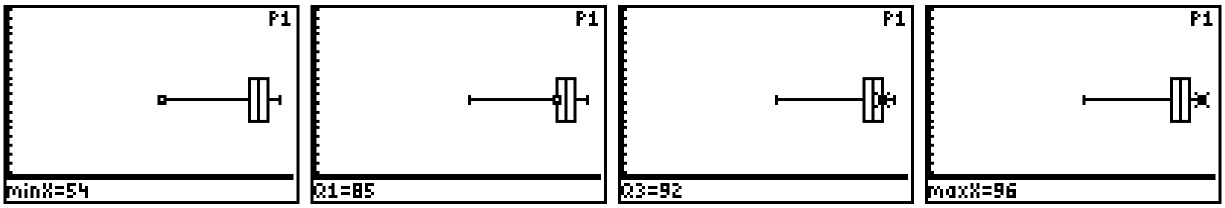


4. Results as follows:



5. Tracing:





6. Try different windows to see what happens.

Instructional activity 2

Scores on the first Physics test are as follows:

Class 1

Student	A	B	C	D	E	F	G	H	I	J
Score	55	64	83	92	100	77	86	95	80	98

Class 2

Student	A	B	C	D	E	F	G	H	I
Score	52	79	71	100	100	76	100	78	76

1. Make a box-and-whiskers plot of each set of data on the same graphics screen. You will have to use two different LISTS and two different STAT PLOTS.

2. Sketch each box-and-whisker, identifying the Min, Q1, Med, Q2, and Max of each. Which class did better?

3. What is the average (mean or median?) score for each class? (When you prompt the calculator to do 1-variable statistics, you *must* follow the prompt with the “place” you stored the statistics.)

4. Does this change your opinion about which class did better?

5. When statistics are quoted to support arguments or positions, what words can be deceiving?

6. Can the wording affect how one perceives the overall picture of “which class did better?”

Instructional activity 3

The left hand is controlled by the right side of the brain, while the right hand is controlled by the left side of the brain. An experiment found a significant difference between males and females pertaining to their ability to identify objects held in their left hand. The test involved 20 small objects that participants were not allowed to see. First they held 10 of the objects, one by one, in their left hands and guessed what the objects were. Then they held the other 10 objects, one by one, in their right hands and guessed what they were. Data gathered are shown in the table below.

Correct Guesses

Females Left hand	Females Right hand	Males Left hand	Males Right hand
8	4	7	12
9	1	8	6
12	8	7	12
11	12	5	12
10	11	7	7
8	11	8	11
12	13	11	12
7	12	4	8
9	11	10	12
11	12	14	11

1. Make a box-and-whisker plot that will allow you to compare the data.

Domain and Range

Organizing topic	A Transformation Approach to Relations and Functions
Overview	Students review domain and range from Algebra I and extend their knowledge.
Related Standard of Learning	AII.9

Objective

- The student will identify the domain and range of a function when given the graph or the equation of the function.

Materials needed

- A "Domain and Range" handout for each student

Instructional activity

1. Review domain and range with the students:
   - What type of function relates the variables?
   - What is the dependent variable?
   - What is the independent variable?
   - What are the constants in the function? What do they mean?
   - What restrictions does the function place on the independent variable?
   - What is a reasonable domain for the function?
   - What is a reasonable range for the function?
2. Give each student a "Domain and Range" handout, showing six problems. Direct the students to sketch a complete graph for the given function in each problem and show the coordinates of any intercepts. If any intercept does not exist, have them explain why.
3. Have the students describe the domain and range for each mathematical situation.

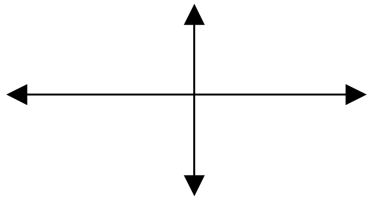
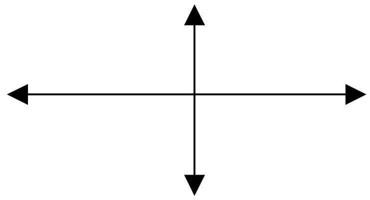
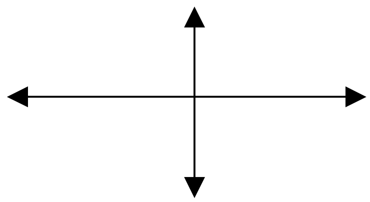
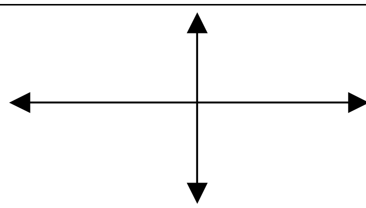
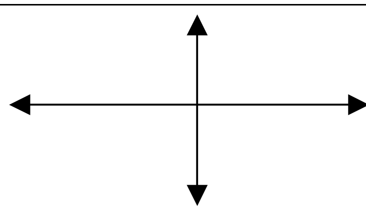
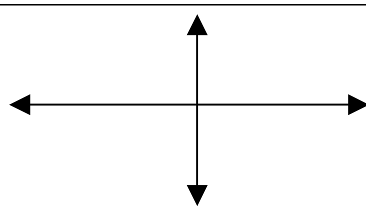
Follow-up/extension

- Have the students compare the ranges of $f(x) = \frac{1}{2}x^2$ and $f(x) = \frac{1}{2}x^2 + 3$.

Homework

- If the function $y = x(5 - x)$ is multiplied by -1, how will the domain and range be affected?

Domain and Range

Function	Graph	Domain and Range
1. $f(x) = \frac{1}{2}x^2$		Domain: Range:
2. $y = x^2 + 3$		Domain: Range:
3. $y = -3x^2$		Domain: Range:
4. $y = x(5 - x)$		Domain: Range:
5. $m(x) = \left(\frac{1}{3}\right)^x$		Domain: Range:
6. $h(x) = 3^b$		Domain: Range:

4. Write a summary comparing the functions. Compare their domains and ranges and their graphs.
5. Describe a practical situation that the functions in #4, #5, and #6 might represent. What restrictions will the situation make on the domain and range of the function? How will the situation affect the graph of the function?

Inverse of a Function

Organizing topic A Transformation Approach to Relations and Functions

Overview Students develop the concept of composition of functions and finding the inverse of a function.

Related Standard of Learning AII.9

Objectives

- The student will find the value of a function for a given element in the domain.
- The student will find the composition of functions algebraically and graphically.
- The student will find the inverse of a function algebraically and graphically.
- The student will explain how composition of functions and finding the inverse of a function affect the domain and range of the functions.
- The student will demonstrate that the exponential and logarithmic functions are inverse functions.

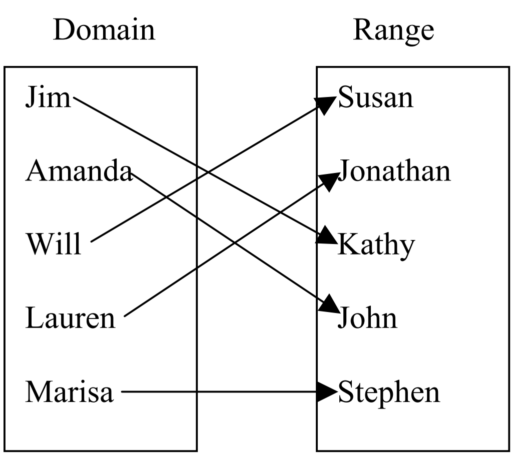
Materials needed

- Graph paper
- Graphing calculators

Instructional activity

Definition of *inverse function:* A function machine, *f*, receives an input, *x*, and outputs the value *f(x)*. The inverse of *f* receives as input the value *f(x)* and outputs *x*.

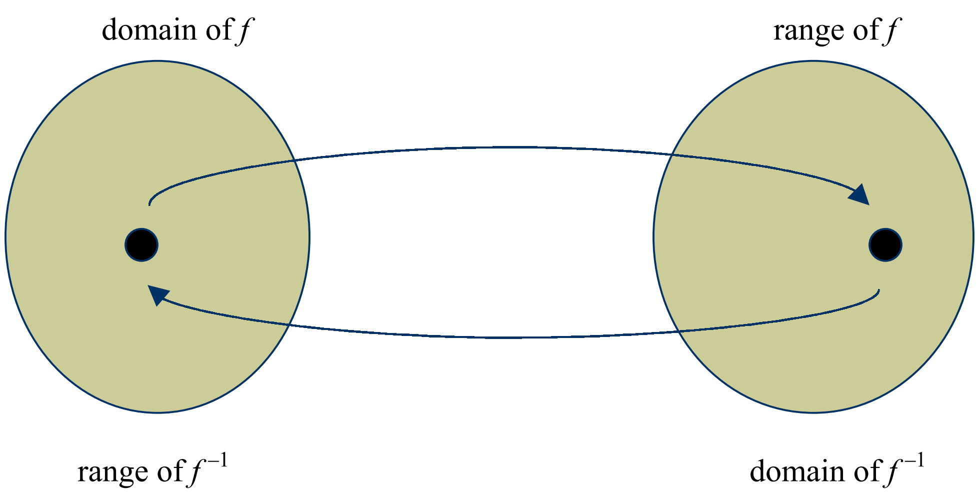
1. Using the definition above, have the students find the inverse of the function below, where the elements of the domain are friends of Monica, and the range consists of the friends' spouses.

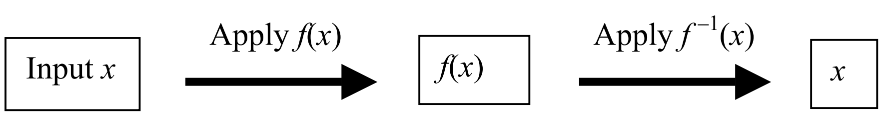


Is the inverse a function?

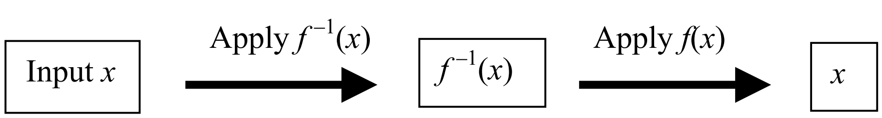
2. Find the inverse of the function {(–3, 27) (–2, –8) (–1, –1) (0, 0) (1, 1) (2, 8) (3, 27)}. Is the inverse a function?
3. Find the inverse of the function {(–3, 9) (–2, 4) (–1, 1) (0, 0) (1, 1) (2, 4) (3, 9)}. Is the inverse a function?

4. How can you tell by looking at a function if its inverse will also be a function?
   - Functions where the unique inputs (values of x from the domain) correspond (map) to unique outputs (values of $f(x)$ from the range) have an inverse that is a function. This is a one-to-one function.
5. What is the vertical line test? Why does it work?
6. Would a horizontal line test tell us anything useful?
   - Graph $y = -0.5(x + 2)^2 + 3$.
   - Graph the horizontal line $y = 2$. Where does the line intersect the parabola?
   - Why does the horizontal line test work?
7. A function $y = f(x)$ is a rule that tells us to do something with the argument, x. $f(x) = 2x$ tells us to multiply the argument by 2. The inverse of f undoes whatever f does. Therefore, $f^{-1}(x) = \frac{1}{2}x$.
8. For a function $y = f(x)$ to have an inverse function, f must be one-to-one. Then for every x in the domain, there is exactly one y in its range. Each y in the range of the function corresponds to exactly one x in the domain. The correspondence from the range of f onto the domain of f is, therefore, also a function. It is this function that is the inverse of f.





$f^{-1}[f(x)] = x$



$f[f^{-1}(x)] = x$

9. Is each of the following functions an inverse?

$h(x) = x^3$ and $h^{-1}(x) = \sqrt[3]{x}$

$h[h^{-1}(x)] = x$ and $h^{-1}[h(x)] = x$

$h(\sqrt[3]{x}) = x$ and $h^{-1}(x3) = x$

$\left(\sqrt[3]{x}\right)^3 = x$ and $\sqrt[3]{x^3} = x$

10. Using your graphing calculator, enter Y1 = x^3 Y2 = $\sqrt[3]{x}$, and set WINDOW so that $-3 \le x \le 3$ and $-2 \le y \le 2$. What do you observe about the graph?
11. Enter Y3 = x. Note the symmetry of the inverse functions with respect to $y = x$. Think about reflections. Use the TABLE function. Notice that if (a, b) is on the graph of f, then (b, a) is on the graph of f^{-1}.
12. Graph the points (–2, –1) (–1, 0) (2, 1) on a set of coordinate axes. Connect the points with line segments. Graph the inverse, and then graph $y = x$.
13. Finding the inverse: The graph of a one-to-one function, f, and its inverse are symmetric with respect to $y = x$. Therefore, we can identify f^{-1} by interchanging the roles of x and y. If f is defined by the equation $y = f(x)$, then f^{-1} is defined by the equation $x = f(y)$. The equation $x = f(y)$ defines f^{-1} implicitly. Solving for y will produce the explicit form of f^{-1} as $y = f^{-1}(x)$.
14. Let $f(x) = 2x + 3$ Is f one-to-one? Yes, it is linear and increasing.

$y = 2x + 3$

$x = 2y + 3$ The variables x and y have been interchanged in the original equation. This equation defines f^{-1} implicitly.

Solving for y,

$$2y + 3 = x$$
$$2y = x - 3$$
$$y = \frac{1}{2}(x - 3)$$
$$f^{-1}(x) = \frac{1}{2}(x - 3)$$ This is the explicit form.

15. What is the domain of f? Of f^{-1}?
16. What is the range of f? Of f^{-1}?
17. Graph Y1 = $f(x) = 2x + 3$ and its inverse, Y2 = $f^{-1}(x) = \frac{1}{2}(x - 3)$, with Y3 = x. Note the symmetry of the graphs with respect to $y = x$.

Follow-up/extension

- Find the inverse of $y = f(x) = x^2$. Because $f(x)$ is not one-to-one on its domain, restrict the domain to $x \ge 0$. Verify the inverse of the function algebraically and graphically, using a restricted domain.

Homework

- Find the inverse of $f(x) = \dfrac{2x+1}{x-1}$, $x \ne 1$. Check your results algebraically and graphically.

Functionality

Organizing topic A Transformation Approach to Relations and Functions

Overview Students review factoring from Algebra I and extend their knowledge into the Fundamental Theorem of Algebra.

Related Standards of Learning AII.1, AII.5, AII.6, AII.10

Objectives

- The student will factor polynomials completely.
- The student will find the zeros of a function algebraically and graphically.
- The student will investigate the relationship between the solutions of a equation, zeros of a function, x-intercepts, and factors of a polynomial.
- The student will identify examples of the field properties of real numbers and the properties of equality that occur while solving equations.
- The student will, given the zeros of a polynomial, write an equation for the polynomial.

Materials needed

- Graphing calculators
- Algeblocks™
- Handouts from Algeblocks™ Unit 8.5 – 8.7 "Multiplying Binomials"
- Handouts from Algeblocks™ Unit 9.7 – 9.11 "Factoring Trinomials"

Instructional activity 1

1. See the Teacher's Resource Binder for Algeblocks™ , and review with students the beginning of Units 8 and 9.
2. Remember that students need to work directly with the blocks. Students also need to draw the pictorial representations and write in symbolic form. Since this activity is a follow-up for the work that students have done with the Algeblocks™ before, it should not take a lot of direct demonstration by the teacher. Students should be ready to work on their own following a short review.

Follow-up/extension

- Create a matching activity of binomials to multiply and trinomials to factor for students to use for practice.

Instructional activity 2

1. Students should preview this activity by solving several linear equations. They may use Algeblocks™, the graphing calculator, and/or symbolic manipulation. The teacher should emphasize using the graphing calculator, locating the x-intercepts of the graph, and identifying the x-coordinate as the zero of the function.
2. Discuss the Fundamental Theorem of Algebra with the students. Show students the graphs of a wide variety of polynomial functions, and have them discuss the possible number of zeros.
3. Allow students to graph polynomial functions and identify the zeros, using the functions of the graphing calculator.

Homework

- Have students explain why a zero of a function is so named.

Practice Solving Quadratics

Organizing topic A Transformation Approach to Relations and Functions

Overview This activity engages students in practicing solving quadratics over C.

Related Standard of Learning AII.6

Objectives

- The student will find the zeros of a function algebraically and graphically.
- The student will investigate the relationship between the solutions of an equation, zeros of a function, *x*-intercepts, and factors of a polynomial.
- The student will, given the zeros of a polynomial, write an equation for the polynomial function.
- The student will identify examples of the field properties of real numbers and the properties of equality that occur while solving equations.

Materials needed

- Two sets of numbers, or cards numbered 1–12, in a container for random drawing
- A “Solving Quadratics” handout for each student

Instructional activity

1. Give each student a copy of the handout.
2. Have each student draw a number at random to determine which problem to do and then return the numbers to the container for the next round of drawing.
3. Have the students work their first problem. After they have completed their problem, have them check their work in pairs before drawing for the next assigned problem.
4. Assign the technique the students must use in solving certain of the problems (optional).

Homework

- Give the students additional problems from the activity, or give the same problems but have the students solve them, using a different technique.

Solving Quadratics

Solve the following quadratics. Show all steps.

1. $2x^2 + 4x + 15 = 0$

2. $5x^2 = 2x - 8$

3. $6x^2 - x + 24 = 0$

4. $15x^2 + 2x + 1 = 0$

5. $9x^2 + 3x + 4 = 0$

6. $3x^2 - 2x + 4 = 0$

7. $3x^2 - 2x + 1 = 0$

8. $2x^2 + 3x = -8$

9. $3x^2 + 4x = -2$

10. $2x^2 - 3x + 5 = 0$

11. $3x(x + 1) = x - 5$

12. $2x^2 + 8 = x$

13. $7x - 13 = x^2$

14. $x^2 + 3x + 5 = 0$

15. $x^2 + 4 = 2x$

Quadratic Applications

Organizing topic A Transformation Approach to Relations and Functions

Overview Through working with problems in context, students develop and apply their understanding of quadratics.

Related Standards of Learning AII.6, AII.10

Objectives

- The student will find the zeros of a function algebraically and graphically.
- The student will investigate the relationship between the solutions of an equation, zeros of a function, x-intercepts, and factors of a polynomial.
- The student will, given the zeros of a polynomial, write an equation for the polynomial function.
- The student will identify examples of the field properties of real numbers and the properties of equality that occur while solving equations.

Materials needed

- Graphing calculators
- A "Quadratic Applications" handout for each student

Instructional activity

1. Give each student a copy of the handout.
2. Divide the class into small work groups.
3. Have the groups solve the problems, and have each student record the work on his/her handout. Tell the students that they will be responsible for explaining their solutions to the class.
4. After the problems are solved, hold a class discussion about the solution to each problem.

Homework

- Have the students complete the homework problems on the handout on their own.
- Allow them to use the graphing calculator to check results and defend answers.

Quadratic Applications

Work with the members of your group to complete the following problems, but record your work on your own sheet. Be sure that you understand each part of your solutions. When the class discusses these problems, any you may be chosen to explain the problem.

Problem 1

Suppose you are standing on a cliff 110 m above the beach. You throw a stone vertically upward at 17 m/s. After reaching its maximum height, the stone falls to the beach, just missing the cliff on the way back down. The height of the stone is given by the equation $H = -4.9T^2 + 17T + 110$, where H is the height in meters of the stone above the beach, and T is the elapsed time in seconds. On Earth the force of gravity is 9.8 m/sec^2 (notice that half of this quantity translates into the lead coefficient for this quadratic).

1. Graph.
2. Which coefficient in the equation is the initial velocity of the stone?
3. How would the equation and the graph change if the initial velocity was 22 m/s? 32 m/s? 12 m/s?
4. Which coefficient is the height of the cliff?
5. How would the graph and the equation change if the height of the cliff were 120 m? 140 m? 100 m? 80 m?
6. How long does it take the stone to hit the beach?
7. When does the stone reach its maximum height?
8. What is the maximum height for the stone?

Heavenly Body	Gravity m/s^2
Sun	**273.0**
Moon	**1.6**
Mercury	**3.5**
Venus	**8.9**
Earth	**9.8**
Mars	**3.7**
Jupiter	**24.9**
Saturn	**10.6**
Uranus	**8.9**
Neptune	**11.7**
Pluto	**0.59**

Suppose you could throw stones upward at 17 m/s from a 110 m cliff on the heavenly bodies listed in the chart on the right.

1. Give the equation to find H for each location.
2. How would the graph of this problem change if you were to use these values?
3. Determine the maximum height of the stone on the moon, the time to reach that height, and the time needed to hit the ground.

Problem 2

A transit company carries about 80,000 riders per day for a fare of $1.25. A survey indicates that if the fare increased, the number of riders will decrease by 360 for every cent of increase.

1. Define the variable, and write an equation for the revenue R in dollars from ticket sales.
2. Expand the equation in #1 into quadratic form.
3. Calculate the fare increase that will result in the greatest revenue for the company.
4. Calculate the fare increase that will result in a 5% increase in revenue.
5. Explain why there are two answers to #4.

Problem 3

A lifeguard has 250 m of rope to set out a rectangular swimming area for children along a lakefront. She can do this in many ways, keeping the length at 250 m, but the area enclosed will vary.

1. Verify that the area enclosed does vary by sketching three possibilities showing the dimensions and area of the rectangular region.

2. Let X meters represent the length of rope perpendicular to the lakeshore. Write an equation for the area $A(X)$ enclosed.
3. Calculate the dimensions of the rectangle that encloses 6,000 m^2 of water.
4. Calculate the maximum possible area of water that can be enclosed.
5. What are the dimensions of the rectangle that enclosed this maximum area?

Problem 4

I. M. Chisov of the USSR set a record in January 1942 for the highest altitude from which someone survived after jumping out of an airplane without using a parachute. He jumped out at 21,980 feet.

6. Write an equation describing his height, H, off the ground at time, T, seconds.
7. Sketch a graph of this fall, using the graphing calculator.
8. State the window you used to enclose this graph.
9. About how long did it take for his fall? Explain how you arrived at this result.

Work the following problems at home on your own. You may use the graphing calculator to check results and defend your answers.

Problem 1

Pop Upp hits a pitch that comes over home plate about 3.5 ft. above the ground. The ball travels towards the outfield on a path described by the equation $H(T) = -.005X^2 + 2X + 3.5$, where X is the distance in feet of the ball from home plate, and H is the height in feet of the ball at any given instant.

1. Sketch the flight of this hit.
2. When is the ball 8 ft. high?
3. A pitcher is standing on the mound about 60 ft. from home plate. How high is the ball when it is directly over his head? (Disregard the height of the pitcher's mound.)
4. When will Pop's hit reach a height of 100 ft.?
5. How far from home plate will the ball be when it hits the ground?
6. Create and answer one additional question about Pop's hit. (Be creative here!)

Problem 2

A juggler throws a ball from her hand at a height of 1 m with an upward initial velocity of 10 m/s.

1. Write an equation to describe the height of the ball over time.
2. How high will the ball be after 1 second? After 2 seconds?
3. Sketch the path of this ball.
4. How high will the ball go?
5. When will the ball return back to "hand level"?
6. When will the ball hit the ground, if the juggler misses her toss?
7. Create and answer one additional question about the juggler's toss.

Venn Diagrams, Set Theory, and Compound Inequalities

Organizing topic	A Transformation Approach to Relations and Functions
Overview	This activity sets the stage for developing the concept of solving equations and inequalities containing absolute value expressions.
Related Standards of Learning	AII.1, AII.4

Objectives

- The student will define *absolute value*.
- The student will evaluate expressions that contain absolute value.
- The student will recognize that the value of the argument for absolute value must be greater than or equal to zero, and explain why.

Materials needed

- A "Venn Diagrams, Number Lines, and Compound Inequalities" handout for each student

Instructional activity

1. Pair students to work through the activity.
2. Discuss each section with the class before proceeding to the next.

Follow-up/extension

- Have the students reflect on the types of problems they have been working during this activity by addressing the following questions:
  - What do you think is the reason for combining these types of problems?
  - What do they have in common?
  - How are they related?

Homework

- Have the students determine the reason that the value of the argument for absolute value must be greater than or equal to zero. Have them explain their answer.

Venn Diagrams, Number Lines, and Compound Inequalities

1. In a bag of 32 balloons, 20 are round, 16 are red, and 11 are both red and round. Draw two intersecting circles (one type of Venn Diagram), label each part with the information from the problem set-up, and use this diagram to explain your answers.
   a. How many balloons were neither red nor round?
   b. How many were round but not red?
   c. How many were red but not round?

2. Local stores contain100 sleds described as follows: 56 are plastic, 56 are red, 37 are full-length, 31 are red plastic, 13 are full-length and red, 17 are full-length and plastic, and 9 are full-length, red, and plastic.
   a. How many sleds fit none of these characteristics?
   b. How many were only plastic?
   c. How many were only red?
   d. How many were only full-length and plastic?

Graph the following on a number line:

3. $3 < x \leq 5$
4. $x < -1$ or $x \geq 5$
5. $x \geq 0$ and $x < 4$
6. $x \geq 4$ and $x < 7$
7. $x < -5$ and $x > 1$
8. $x \geq 4$ and $x > 7$
9. $x \geq 4$ or $x > 7$
10. $x > -1$ or $x < 3$

Picture on a number line the real numbers that satisfy each of the following:

11. $|x| = 5$
12. $|x| = 0$
13. $|x| = \leq 8$
14. $|x| \geq 6$
15. $|x| \geq -1$
16. $|x| = \leq 0$
17. $|x - 3| \leq 5$
18. $|x + 3| \leq 6$
19. $|x - 4| \geq 3$
20. $|2x - 5| = \leq 16$

As an extension of the problems above, reflect on the *types* of problems you have been working during this activity.

1. What do you think is the reason for combining these types of problems?
2. What do they have in common?
3. How are they related?

For homework, answer the following question, and explain your answer:

4. What is the reason that the value of the argument for absolute value must be greater than or equal to zero?

Solving Compound Inequalities and Absolute Value Equations and Inequalities

Organizing topic A Transformation Approach to Relations and Functions

Overview This activity engages students in practice and refinement of their ability to solve equations and inequalities involving absolute value.

Related Standards of Learning AII.1, AII.4

Objectives

- The student will solve absolute value equations algebraically and graphically.
- The student will identify examples of field properties of real numbers and the properties of equality that occur while solving equations.
- The student will solve absolute value inequalities algebraically and graphically.
- The student will identify examples of the properties of inequality and order that occur while solving inequalities.

Materials needed

- Handouts containing the questions for each work station, one handout for each student
- Large index cards containing the answers and full solutions
- Timer to determine when to change stations

Instructional activity

Note: This activity is designed as a review and practice session on these topics, with the students assisting each other through the review process. Students will determine the real value of the activity by the seriousness with which they work.

1. Set up six workstations in the room, each station having half-sheets of paper containing the set of problems designated for that station. Be sure that each station has enough sheets for every student in the room to have one. Place a large index card at each station with the answers and full solutions to the problems.
2. Place students into groups at the six stations.
3. Give each group 7 to10 minutes per station to complete the problems found there, working together and checking answers.
4. Set the timer so that the groups know when to stop working and rotate to the next station.

STATION #1 PROBLEMS

Solve and graph:

1. $3(x - 2) \geq 5$
2. $6 - 4x \leq 10$
3. $\frac{x}{3} + 5 \leq 7$
4. $-5 \leq 2x + 1 \leq 3$
5. $7x - (3 - 2x) \geq x - 3$

STATION #2 PROBLEMS

Solve and graph:

1. $3x + 6 \geq 4$ or $6x - 2 \geq 8$
2. $4x + 2 \geq 10$ or $-4x - 2 \geq 10$
3. $\frac{x}{-4} + 6 > 10$ and $\frac{2x - 1}{3} > 2$
4. $-5x < 10$ or $4x < 16$
5. Graph: $x > 5$ and $x > 9$

STATION #3 PROBLEMS

Solve and graph:

1. $|2x + 5| = 3$
2. $|5 - x| + 4 = 10$
3. $-2|x + 6| = -10$
4. $|2x - 3| \leq 5$
5. $|11 - 3x| + 6 > 10$

STATION #4 PROBLEMS

Solve and graph:

1. $3|2x - 2| + 8 = 23$
2. $|3x - 5| \geq 4$
3. $4|-x + 4| < 12$
4. $|5 - (2x + 3)| \geq 4$
5. $|2 - x| < 9$

STATION #5 PROBLEMS

Write inequalities for each of the following:

1. You have $20,000 available to invest in stocks A and B. Write an inequality stating the restriction on A if at least $3,000 must be invested in each stock.

2. The largest egg laid by any bird is that of an ostrich. An ostrich egg can reach 8 inches in length. The smallest egg is that of a hummingbird. Its eggs are approximately 0.4 inches in length. Write an inequality that represents the variety that occurs in the lengths of bird eggs.

3. On Pennsylvania's interstate highway the speed limit is 55 mph. The minimum speed limit is 45 mph. Write a compound inequality that represents the legal speed limits.

4. You have $60 and a coupon that allows you to take $10 off any purchase of $50 or more at the department store. Write an inequality that describes the possible retail value of the items you can buy if you use this coupon. Write an inequality that describes the different amounts of money you can spend if you use the coupon.

5. Develop a context for the inequality $0 < 2x - 5 < 30$

6. Develop a context for the inequality $|x - 7| > 12$

STATION #6 PROBLEMS

1. Set up the inequality and solve: Maria has grades of 79, 67, 83, and 90 on four tests. What is the lowest grade she can make on the fifth test if she wants to have an average of at least 82?

2. Tickets at the county fair cost $1.50 for the first ticket and $1.25 for each additional ticket. If you have $10.25 to spend on tickets, how many can you purchase in all? Be sure to define the variable you are using and state the inequality.

3. Most fish can adjust to a change in water temperature of up to 15º F if the change isn't too sudden. Suppose lake trout live comfortably in water 58º F. Write an inequality that represents the range of temperatures at which the lake trout can survive. Write an absolute value inequality that represents the same information.

4. Create a context for $\left|\frac{x-5}{2}\right| \le 10$.

Conic Sections Practice

Organizing topic A Transformation Approach to Relations and Functions

Overview This activity gives students the opportunity to apply the definitions of the conic sections in varying situations.

Related Standard of Learning AII.18

Objectives

- The student will recognize the graphs of the conic sections defined as any figure that can be formed by slicing a double cone
- The student will, given an equation, identify the conic section and graph it transformationally.

Materials needed

- A "Conic Sections Practice" handout for each student

Instructional activity

- Place students into pairs to work together through the problems on the handout.

Sample resources

Mathematics SOL Curriculum Framework
http://www.pen.k12.va.us/VDOE/Instruction/Math/math_framework.html

SOL Test Blueprints

Released SOL Test Items http://www.pen.k12.va.us/VDOE/Assessment/Release2003/index.html

Virginia Algebra Resource Center http://curry.edschool.virginia.edu/k12/algebra

NASA http://spacelink.nasa.gov/index.html

The Math Forum http://forum.swarthmore.edu/

4teachers http://www.4teachers.org

Appalachia Educational Laboratory (AEL) http://www.ael.org/pnp/index.htm

Eisenhower National Clearinghouse http://www.enc.org/

Conic Sections Practice

Work with a partner to complete this worksheet, but each of you must individually record your answers. At the top of your paper, be sure to write your name and your partner's name.

Name: ______________________________ **Partner's Name:** ___________________________

I. Give a written description of the graph of each of the following. Be as detailed as possible.

1. $x^2 + y^2 = 49$
2. $4x^2 + y^2 > 100$
3. $x^2 + 9y^2 = 0$
4. $x^2 - y^2 < 25$
5. $x^2 + y = 16$
6. $x - y^2 > 10$
7. $4x^2 - 3y^2 = 24$

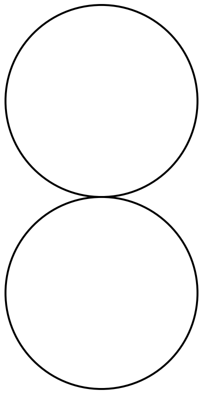
II. Given $3x^2 + 4xy + y^2 = 0$,

8. Which conic does this equation describe?
9. Factor the left side of this equation, yielding __________________________ .
10. Complete the solution of the equation formed in #9.
11. Graph all points satisfying the given equation. This is called the "degenerate" form of a conic. Which conic? _____________________

III. Sketch the graph of each equation:

12. $y^2 = x^2 + 16$
13. $4x^2 + 25y^2 = 100$
14. $x + y^2 = 9$
15. $y - x^2 = -4$

IV.



In a skating competition, figure skaters must complete a compulsory figure eight (pictured at left) as part of the scoring process. Each skater must create on the ice each of the circles with a radius twice the height of the skater. Assume that a skater begins at the point where the circles are tangent, and call this the origin of the coordinate system. If the skater is 5 ft. 7 in. tall, write a sentence to describe the following:

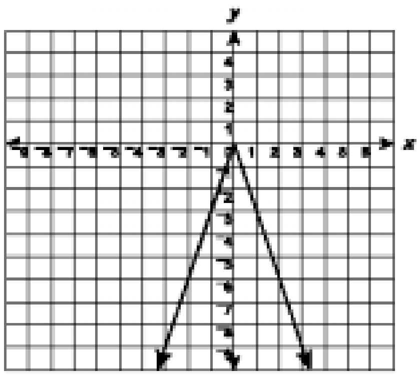
16. the points on the upper circle

17. the points on the lower circle

18. the points in the interior of the upper circle.

Sample assessment

The graph below is an example of which type of function?



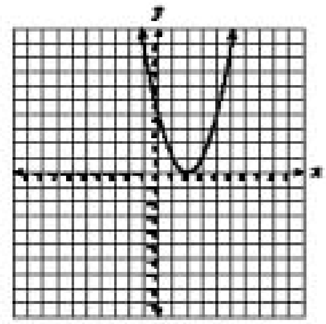
A Absolute value
B Exponential
C Linear
D Quadratic

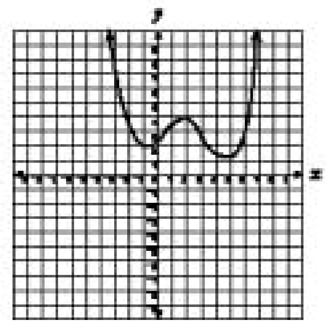
If $f(x) = 5x^2 - 7$, what is $f(-3)$?
F –52
G –22
H 38
J 45

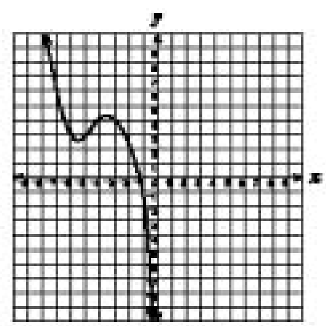
Which value is *not* a zero of $P(x) = x^3 = 3x^2 - x - 3$?
A 1
B –1
C 3
D –3

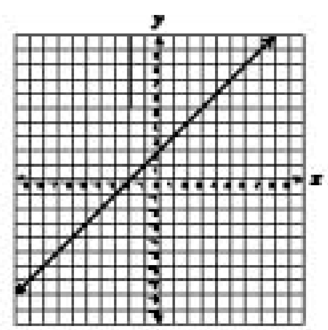
The amount of interest, I, owed on a loan varies directly with the length of time, t, of the loan. If k is the constant of proportionality, which formula represents this relationship?
A $I = kt$
B $I = \dfrac{k}{t}$
C $t = kI$
D $t = \dfrac{k^2}{I}$

Which graph could represent a polynomial function with *no* real zeros?

F


G


H


J


Given $f(x) = x^2 - 1$ and $g(x) = x^2$, which of the following expressions represents $g(f(x))$?

A $x^2\sqrt{x^2 - 1}$

B x

C $\sqrt{x^4 - 1}$

D $(x^2 - 1)^2$

Which of the following represents the solution to $|x| = 7$?

F $x = 7$

G $x = 0$

H $x = -7$

J $x = -7$ or $x = 7$

What are the solutions to $y^2 - 4y + 4 = 36$?

A $y = -4$ or $y = 8$

B $y = 4$ or $y = 8i$

C $y = \pm 4$

D $y = \pm 4i$

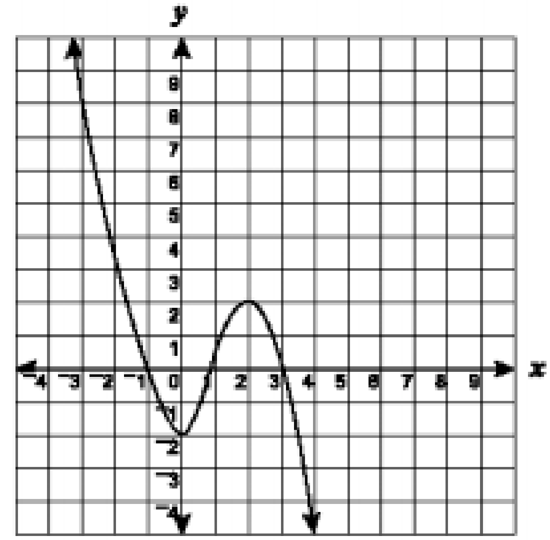
Which of the following functions has x-intercepts at –2 and 1?

A $y = x^2 - x - 2$

B $y = x^2 + x - 2$

C $y = x^2 - 2x + 1$

D $y = 2x - 1$



The polynomial function above shown apparently has zeros at

F –1 and 2

G –1, 0.7, and 3

H –2

J 1, –0.7, and –3

Organizing Topic Sequences and Series

Standard of Learning

AII.16 The student will investigate and apply the properties of arithmetic and geometric sequences and series to solve practical problems, including writing the first n terms, finding the n^{th} term, and evaluating summation formulas. Notation will include Σ and a_n.

Essential understandings, knowledge, and skills	Correlation to textbooks and other instructional materials
• Find the next term in a sequence by looking for a pattern.	____________________
• Find the n^{th} term of an arithmetic sequence and find the position of a given term in an arithmetic sequence.	____________________
• Find arithmetic means.	____________________
• Differentiate between a sequence and a series.	____________________
• Find the sum of an arithmetic series.	____________________
• Find specific terms in an arithmetic series.	____________________
• Use sigma (Σ) notation to denote sums.	____________________
• Compare and contrast arithmetic and geometric sequences.	____________________
• Find the n^{th} term of a geometric sequence and the position of a given term in a geometric sequence.	____________________
• Find geometric means.	____________________
• Find the sum of a geometric series.	____________________
• Find specific terms in a geometric series.	____________________

Guess My Rule!

(From *Problems with a Point.* January 6, 2001. ©EDC 2001)

Organizing topic	Sequences and Series
Overview	Students explore number patterns.
Related Standard of Learning	AII.16

Objective

- The student will find the next term in a sequence by looking for a pattern.

Materials needed

- A "Guess My Rule!" handout for each student

Instructional activity

- Have the students work through the problems in part I on the "Guess My Rule!" handout.

Follow-up/extension

- Give the students additional practice in finding the next term in a sequence by having them work the problems in part II.

Guess My Rule!

Part I
The sequence 1, 4, ... might be continued in many ways. Here are four possibilities and the rules that they follow:

1, 4, 7, 10, ... $a_n = 3n - 2$, an arithmetic sequence

1, 4, 16, 64, ... $a_n = 4^{n-1}$, a geometric sequence

1, 4, 27, 256, ... $a_n = n^n$

1, 4, 1, 4, ... $a_n = 4^?$ (This solution has been left unfinished. You may figure out the missing part as you complete the problems below!)

> Each number in the sequence is a_n, with n representing the position in the sequence. Another way to write the first formula is $f(n) = 3n - 2$.

For each sequence below, think of *at least* two rules that might describe the sequence. State the rules, and show how the sequence continues. Express your rules algebraically if you can.

1. 0, 1, ...

2. 1, –1, ...

3. 2, 1, 2, ...

4. 10, 100, ...

5. 10, 101, ...

6. $\frac{1}{3}, \frac{1}{2}, ...$

Compare your findings with those of your classmates. Compile a class list of solutions for each item.

Part II
For each sequence below, think of *at least* two rules that might describe the sequence. State the rules, and show how the sequence continues. Express your rules algebraically if you can.

7. 5, 6, ...

8. 5, 7, ...

9. 5, 10, ...

10. 5, 25, ...

11. 5, –5, ...

12. $5, \frac{1}{5}, ...$

Inventing a Formula for Arithmetic Sequences

(From *Problems with a Point.* February 2, 2001. ©EDC 2000)

Organizing topic Sequences and Series

Overview Students have an opportunity to invent formulas for arithmetic sequences.

Related Standard of Learning AII.16

Objectives

- The student will find the n^{th} term of an arithmetic sequence and find the position of a given term in an arithmetic sequence.
- The student will find specific terms in an arithmetic series.

Materials needed

- Computer lab or computer with projection device suitable for viewing by all students
- An "Inventing a Formula for Arithmetic Sequences" handout for each student

Instructional activity

1. In the computer lab, have students go to the Drexel University's Math Forum Web site and read and take notes on the following articles:
    - "Finding a Term of an Arithmetic Series" http://mathforum.org/library/drmath/view/56871.html
    - "Circle in n Sectors" http://mathforum.org/library/drmath/view/54854.html
    - "Formula for a Sequence" http://mathforum.org/library/drmath/view/56933.html
2. Have the students work through the problems shown on the "Inventing a Formula for Arithmetic Sequences" handout.

Inventing a Formula for Arithmetic Sequences

Here are the first five terms of an arithmetic sequence: 4, 7, 10, 13, 16, ...

> In an *arithmetic sequence*, the difference between any two successive numbers is the same.

> You can figure out nos. 3, 4, and 5 without knowing the terms in between, and without using any special formula.

1. What are the next three terms?
2. How much greater is the second term than the first?
3. How much greater is the third term than the first?
4. How much greater is T_4 (the fourth term) than T_1?
5. How much greater is T_7 (the seventh term) than T_4?
6. What is the value of T_{100}?

Here are the first three terms of another arithmetic sequence: 53, 49.2, 45.4, ...

> The *rate of change* is one name for the constant difference between any two successive terms of an arithmetic sequence. Pay attention to the *sign* of the rate of change.

7. What is the rate of change, $T_2 - T_1$, in this sequence?
8. What are the next three terms?
9. Compute $T_6 - T_1$. Explain why it is five times $T_2 - T_1$.
10. Explain how it is possible to compute $T_8 - T_1$ without computing T_8. Compute $T_8 - T_1$.
11. Compute $T_{26} - T_{21}$ without knowing either T_{26} or T_{21}.
12. What is the value of T_{100}?

Here is yet another arithmetic sequence: a, $a + b$, $a + 2b$, ...

13. What is the rate of change in this sequence?
14. What are terms T_1, T_2, T_3, T_4, and T_5?
15. Compute T_{100}.
16. Compute $T_{6,000}$.
17. Compute T_n.
18. Compute $T_9 - T_1$.
19. Compute $T_{100} - T_1$.
20. Compute $T_{100} - T_{83}$.
21. Compute $T_n - T_k$.

22. Explain in words (or as a formula) how to find T_n of *any* arithmetic sequence if you know T_1 and b (rate of change).

What Are Arithmetic Sequences and Series For?

(From *Problems with a Point.* February 1, 2001. ©EDC 2000)

Organizing topic Sequences and Series

Overview Students examine sequences and series in context.

Related Standard of Learning AII.16

Objectives

- The student will differentiate between a sequence and a series.
- The student will find the sum of an arithmetic series.

Materials needed

- A "What Are Arithmetic Sequences and Series *For*?" handout for each student

Instructional activity

- Have the students work through the problems on the handout.

Follow-up/extension

- In the computer lab, have students go to the Math Forum at Drexel University http://mathforum.org/library/drmath/view/56872.html and read and note the article, "Finding the 1000th Term in a Sequence."
- Show the students that the expression formed by adding the terms of an arithmetic sequence is called an *arithmetic series*. The sum of the first n terms of an arithmetic series is called S_n. To find a rule for S_n, you may write S_n in two different ways:
  - $S_n = a_1 + (a_1 + d) + (a_1 + 2d) + ... + a_n$
  - $S_n = a_n + (a_n - d) + (a_n - 2d) + ... + a_1$
- Have the students find a formula for the sum of the first n terms of an arithmetic series.

What Are Arithmetic Sequences and Series *For*?

When a ball is thrown straight up in the air, it starts out fast, gradually slows down until it reaches its maximum height, and then starts coming down. During each unit of time, the ascending ball goes up less than it did during the previous unit. Remarkably, the amount of decrease from one unit of time to the next is constant!

A *change in speed* can be described as a change in distance traveled during a unit of time.

The amount of change depends on gravity, which is different on different planets.

For example, suppose the ball moves straight up 52 inches during the first tenth of a second after it is thrown. During the next $\frac{1}{10}$ sec., it won't rise as much: suppose it rises only 48 inches — that is, four inches less than it did in the first $\frac{1}{10}$ sec. Then, during each tenth of a second, it will again rise four inches less than it did during the tenth of a second before. One might ask these two questions about such a situation:

- How long will it be before the ball starts coming back down?
- How high will it go?

In this problem, you are creating an *arithmetic* sequence — a sequence of numbers whose terms differ by a constant amount (in this case, –0.8) from one to the next.

One way to figure out answers to questions like these is to list how far the ball travels each tenth of a second, as shown in the table below.

Time after release from hand	1st tenth of a sec.	2nd tenth of a sec.	3rd tenth of a sec.	...
Distance traveled during that tenth of a second	52 in.	48 in.	44 in.	...

1. Create a table like this, and fill it in for each tenth of a second for the first 16 tenths of a second.
2. What total distance does the ball travel upward during the first 0.2 sec.? (Here you are *summing* an arithmetic series.)
3. How far does the ball travel during the first second?
4. At what time does the ball no longer go up? (Here, you are finding which term in an arithmetic sequence has a particular value.)
5. From the table, how high does the ball go at its highest point? (Again, to solve this, you are *summing* a series.
6. What do the negative numbers in your table mean? (Hint: Think about where the ball is and what it is doing just a moment earlier. Think what must happen next. Then explain the number [both its *size* and its *sign*] in terms of what the ball is doing.)
7. From the time the ball reaches its highest point, how long will it be before the ball reaches your hand again? (Try to figure this out without extending your table. Hint: Compare the distances traveled in the two-tenths of a second just before and after the very top of the ball's trip.)

Geometric Sequences and Series

Organizing topic Sequences and Series

Overview Students gather information about sequences and series from an Internet source.

Related Standard of Learning AII.16

Objective

- The student will compare and contrast arithmetic and geometric sequences.
- The student will find the n^{th} term of a geometric sequence and the position of a given term in a geometric sequence.
- The student will find geometric means.
- The student will use sigma (Σ) notation to denote sums.

Instructional activity

1. In the computer lab, have students go to the Math Forum at Drexel University and read and note the following articles:
   - "Arithmetic and Geometric Progressions"
     http://mathforum.org/library/drmath/view/56925.html
   - "Summation Notation and Arithmetic Series"
     http://mathforum.org/library/drmath/view/56985.html
   - "Formula for the Nth Term in a Geometric Sequence"
     http://mathforum.org/library/drmath/view/56957.html
2. Have the students solve the following problems:

   You buy a new computer for $1,200. The value of the computer decreases by 23% each year.
   - Write a rule for the average annual value of the computer, a_n (in dollars), in terms of the year. Let n = the current year.
   - In about how many years will the value of the computer fall to $200?

Follow-up/extension

- What do the variables in the general rule for a geometric sequence represent?
- How can you determine the common ratio of a geometric sequence from two consecutive terms?

Homework

- Journal: Compare and contrast arithmetic and geometric sequences.

Geometric Series

(From *Problems with* a *Point.* February 2, 2001. ©EDC 2000)

Organizing topic Sequences and Series

Overview Students develop the idea of a series as a sum.

Related Standard of Learning AII.16

Objective

- The student will differentiate between a sequence and a series.
- The student will find the sum of a geometric series.
- The student will find specific terms in a geometric series.

Materials needed

- A "Geometric Series" handout for each student
- One full-size sheet of paper for each student

Instructional activity

- Have the students work through the problems on the "Geometric Series" handout, which details the paper-ripping idea from Dennis Meyer of Becker, MN High School.

Geometric Series

What would you do if you walked into class one day, and found

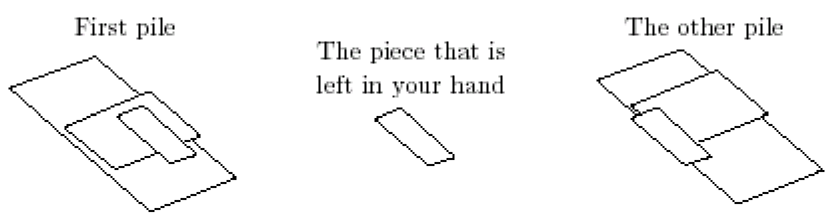
$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}+\frac{1}{729}+\frac{1}{3^7}+\frac{1}{3^8}+\frac{1}{3^9}+\frac{1}{3^{10}}+\frac{1}{3^{11}}+\cdots$$

written on the board, and the teacher asked you to calculate the sum?

a. Get out your calculator as fast as possible.
b. Get out of class as fast as possible.
c. Become hysterical.
d. Start tearing your homework into shreds.
e. None of the above.

If you would use scrap paper instead of your homework, answer d might actually help you! Here's how:

1. Using a full-size, blank sheet of paper, work through the following problem:
   a. Rip the paper roughly into thirds. (The tearing doesn't have to be precise.) Think of the whole paper as representing 1 and each "third" of it as representing $\frac{1}{3}$. Start two piles of paper on the table by placing one of the thirds to your left and another third in a separate pile to your right.
   b. Now repeat the process with the piece of paper that remains in your hand (the last third). Rip it into three parts (ninths of the whole piece), and place one of the ninths on the left-hand pile and another on the right-hand pile.
   c. Once more, rip the part that remains in your hand into three equal pieces. Place one of these on the left-hand pile and another on the right-hand pile.
   d. What fraction of the original paper is now left in your hand?



   e. Your piles should now look roughly like the ones in the diagram above. Repeat this process for several more steps, and write (using pictures if you wish) a description of what you have done.
   f. How many repetitions of this process — ripping the piece in your hand into thirds and placing two of the resulting parts in the piles — did you do before you stopped? What fraction of the original paper was left in your hand?
   g. Is it *theoretically* possible to carry this process far enough so that you'd have less than 1% of the original paper in your hand? Less than 0.1%? Less than 0.01%? Explain your answer. Why does this problem ask, "Is it *theoretically* possible...", instead of, "Is it possible..."?
   h. You constructed the two piles on the table to be identical. *Roughly* how much of the original sheet would each pile represent after repeating the ripping-placing process eight times?
   i. In your own words, tell how the ripping-and-piling-up process represents the sum $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots$

2. Adapt the paper-ripping method to help you think about the following sum. Find the sum, and explain how you found it.

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\frac{1}{1024}+\frac{1}{4^6}+\frac{1}{4^7}+\frac{1}{4^8}+\frac{1}{4^9}+\frac{1}{4^{10}}+\frac{1}{4^{11}}+\cdots$$

3. The paper-ripping method can help you find the following sums, too, but you'll need to think carefully about how many piles you want to create, what you will put in each pile, and what remains in your hand to rip again. Find both sums, and explain your methods and results.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\frac{1}{256}+\frac{1}{512}+\frac{1}{1024}+\frac{1}{2^{11}}+\cdots$$

$$\frac{2}{5}+\frac{2}{25}+\frac{2}{125}+\frac{2}{625}+\frac{2}{5^5}+\frac{2}{5^6}+\frac{2}{5^7}+\frac{2}{5^8}+\frac{2}{5^9}+\frac{2}{5^{10}}+\frac{2}{5^{11}}+\cdots$$

4. The following sum is quite impractical to do with paper, but you can *imagine* doing it with paper. Find the following sum, and explain how you did it. (The world of your imagination can be so much richer than the world of things!)

$$\frac{1}{100^1}+\frac{1}{100^2}+\frac{1}{100^3}+\frac{1}{100^4}+\frac{1}{100^5}+\frac{1}{100^6}+\frac{1}{100^7}+\cdots$$

5. Answer the following two questions about the sums in the charts:

a. Summarize what you know about these sums:

Series	Sum
$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots$	
$\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots$	
$\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\cdots$	
$\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\cdots$	
$\vdots$	$\vdots$
$\frac{1}{100}+\frac{1}{100^2}+\frac{1}{100^3}+\cdots$	

b. From the pattern, conjecture a rule about $\frac{1}{a}+\frac{1}{a^2}+\cdots$.

Series	Sum
$\frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^3}+\cdots$	

6. Here are three more sums. For each one of them, find a paper-ripping method that leads to an answer, and explain your method and how it led to the answer.

$$\frac{1}{10^1}+\frac{1}{10^2}+\frac{1}{10^3}+\frac{1}{10^4}+\frac{1}{10^5}+\frac{1}{10^6}+\frac{1}{10^7}+\cdots$$

$$\frac{3}{10^1}+\frac{3}{10^2}+\frac{3}{10^3}+\frac{3}{10^4}+\frac{3}{10^5}+\frac{3}{10^6}+\frac{3}{10^7}+\cdots$$

$$\frac{9}{10^1}+\frac{9}{10^2}+\frac{9}{10^3}+\frac{9}{10^4}+\frac{9}{10^5}+\frac{9}{10^6}+\frac{9}{10^7}+\cdots$$

7. Generalize the rule you developed in problem 10 so that it helps you with sums like the following:

$$\frac{b}{a}+\frac{b}{a^2}+\frac{b}{a^3}+\frac{b}{a^4}+\cdots$$

Hints

- Hint to problem 1d: One-third of one-ninth is in your hand.
- Hint to problem 1g: Things that are possible in *principle* are not always possible in *practice*.
- Hint to problem 1h: After eight repetitions, what fraction of the paper is left in your hand? What does that mean about how much of the original sheet each pile would represent?
- Hint to problem 2: Make separate piles as you did in problem 1. How many piles will you need?
- Hint to problem 3: One way to get $\frac{2}{5}$ into a pile is to combine two piles, each of which has $\frac{1}{5}$.
- Hint to problem 4: How many piles would you need?
- Hint to problem 6: Combining more than one pile may help with some of these.
- Hint to problem 7: Use problem 6, in which the numerator is not always 1, to generalize your rule from problem 5.

Follow-up/extension

- The following two sums are quite different. Maybe paper-ripping can help solve them, maybe not. Either explain why the paper-ripping method doesn't appear to help with these sums, or find an interpretation of the paper-ripping method that *does* give you some insight into these problems.

$$\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\frac{1}{6!}+\frac{1}{7!}+\frac{1}{8!}+\cdots$$

$$\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{8^2}+\cdots$$

Exploring Exponential Functions

Organizing topic Sequences and Series
Overview Students extend sequence concepts.
Related Standard of Learning AII.16

Objective
- The student will find the n^{th} term of a geometric sequence and the position of a given term in a geometric sequence.

Materials needed
- Graphing calculators
- A sheet of notebook paper for each student

Instructional activity
1. Have the students fold a rectangular sheet of notebook paper in half, noticing that the fold divides the paper into two rectangles. Then have them fold the paper in half again and open the paper to observe the four rectangles formed by the folds. Have them refold their paper and then fold it in half a third time. Have them use a table similar to the one below to record the number of rectangles formed by each number of folds. Direct them to continue folding until they cannot make another fold.

Number of Folds	Number of Rectangles
0	1
1	2
2	4
3	
4	
5	

2. Have the students identify the variables for this relationship and describe the domain and range for this situation. Ask them to describe how the number of rectangles changes as the number of folds increases.
3. Tell the students to express symbolically the relationship between the variables.
4. Ask, "If your paper had 128 rectangles, how many folds would have been made?" Ask them to explain their answer.
5. Have them describe a graph of the data.
6. Have the students describe the relationship between the original number of rectangles and the number of rectangles after one fold. Have them describe the relationship between the number of rectangles after one fold and the number of rectangles after two folds.
7. Ask, "What do you think the relationship will be between the number of rectangles after two folds and the number of rectangles after three folds?"
8. Have the students describe the pattern they notice in the number of rectangles as the number of folds increases.
9. Ask, "What would you have to do to the original number of rectangles to get the second number of rectangles?

10. Ask, "How do the exponents relate to the values in your table?"
11. Have the students suppose that they could continue folding the paper and extend their table to include 10 folds. How many rectangles would there be?

Follow-up/extension

- Have the students describe how the situation would be different if the paper had been folded into thirds each time instead of halves.
- Have the students suppose that they began with a square sheet of paper that measured two feet on a side. What is the relationship between the number of folds and the area of the rectangle after each fold?

Homework

- Have the students explain how the graph of the function $A = 4\left(\frac{1}{2}\right)^n$ would be different from the graph of the original function, $A = 2^n$.

Sequence and Series Application Problems

Organizing topic Sequences and Series

Overview This activity gives students application problems through which to build an understanding of sequences and series.

Related Standard of Learning AII.16

Objectives

- The student will find the n^{th} term of an arithmetic sequence and find the position of a given term in an arithmetic sequence.
- The student will find the sum of an arithmetic series.
- The student will find the n^{th} term of a geometric sequence and the position of a given term in a geometric sequence.
- The student will find the sum of a geometric series.
- The student will find specific terms in a geometric series.

Materials needed

- Graphing calculators
- An "Applications of Sequences and Series" handout for each student

Instructional activity

1. Divide students into groups of no more than three, and have the groups assign a number (1, 2, or 3) to each of its members.
2. Give each student an "Applications of Sequences and Series" handout. All group members must work together to complete this activity, but each member should individually record all work to be turned in. This work must include appropriate formulas, reasons for choosing, calculations, and the final answer.
3. Choose students at random to present and explain problems to the entire class. Number the groups, and draw numbers to determine the order of groups presenting. Allow groups to choose which problem they want to present, but have each group determine by drawing numbers 1, 2, and 3 who will present.
4. Provide students with the formulas for working with arithmetic and geometric sequences and series but give no explanations of when to use them nor the meaning of the symbols (optional).

Applications of Sequences and Series

Your group will determine which problem(s) to present to the class. The order of these presentations will be randomly determined by draw, and a group member will also be chosen by random draw to explain how the problem was solved.

1. A runner begins training by running 5 mi. one week. The second week she runs a total of 6.5 mi. The third week she runs 8 mi. Assume this pattern continues.
   a. How far will she run in the tenth week?
   b. At the end of the tenth week what will be the total distance she has run since she started training?
2. A superball is dropped from a height of 2 m and bounces 90% of its original height on each bounce.
   a. When it hits the ground for the eighth time, how far has it traveled?
   b. How high off the floor is the ball on the eighth bounce?
3. A snail is crawling straight up a wall. The first hour it climbs 16 inches, the second hour it climbs 12 inches. Each succeeding hour it climbs only three-fourths the distance it climbed the previous hour. Assume the pattern continues.
   a. How far does the snail climb during the seventh hour?
   b. What is the total distance the snail has climbed in seven hours?
4. Suppose on Jan.1 you deposit $1.00 in an empty piggy bank. On Jan.8 you deposit $1.50; on Jan. 15 you deposit $2.00; and each week thereafter you deposit $0.50 more than the previous week.
   a. What kind of sequence do these deposits generate?
   b. What amount should you deposit in the 52^{nd} week?
   c. What is the total in the piggy bank at the end of these 52 weeks?
5. Carla's Clothing Shop opened eight years ago. The first year she made $3,000 profit. Each year thereafter her profits were about 50% greater than the previous year.
   a. How much profit did Carla earn during the 18^{th} year of business?
   b. What was the total amount earned over her first 18 years?
6. A ball on a pendulum moves 50 cm on its first swing. Each succeeding swing it moves 0.9 the distance of the previous swing.
   a. Write the first six terms of the sequence generated.
   b. Assuming the pattern continues, how far will the ball travel before coming to rest?
7. Find the sum of the odd integers from 25 to 75.

Extension

Solve each of the following problems individually, explaining how you arrived at your solution. You may use words only, formulas with explanations, or clear diagrams with explanations. Carefully and thoroughly explain how you reached each answer.

1. A person just fitted for contact lenses is told to wear them only 2 hours the first day and to increase the amount of time by 20 minutes each day. After how many days will the person be able to wear the contacts for 14 hours?

2. A wooden ladder is wider at the bottom than at the top and has 10 equally spaced rungs (steps). The bottom rung is 22 inches long and the top rung is 14 inches long. Find the total number of inches of material used to make all the steps.
3. A post is driven into the ground. The first strike drives the post 30 inches into the ground. The next strike drives the post 27 inches into the ground. Assume these distances form a geometric sequence.
    a. What is the total distance the post is driven into the ground after 8 strikes (to the nearest inch).
    b. What is the maximum distance the post could be driven?
4. Poor Mark! His dog ate his math homework. He remembered that the first three terms of his arithmetic sequence were 4, 8, and 12; and the last two were 356 and 360. How many terms were there in this sequence?

Infinite Geometric Series

Organizing topic Sequences and Series

Overview Students collect data to support the concept of infinite geometric series.

Related Standard of Learning AII.16

Objectives

- The student will find the n^{th} term of a geometric sequence and the position of a given term in a geometric sequence.
- The student will find the sum of a geometric series.
- The student will find specific terms in a geometric series.

Materials needed

- Computer lab or computer with projection device suitable for viewing by all students
- An "Infinite Geometric Sequences" handout for each student

Instructional activity

1. In the computer lab, have students go to the Math Forum at Drexel University http://mathforum.org/library/drmath/view/56960.html and read and note the article, "Geometric Sequences and Series." Have them also read and note the article, "Sum of an Infinite Series" at http://mathforum.org/library/drmath/view/55657.html.
2. Have the students work alone or in pairs on this activity.
3. Teacher Notes: This activity is set up for students to use only the graphing calculator. However, teachers may use different kinds of balls (e.g., tennis balls, basketballs, softballs, ping pong balls, golf balls) and allow students to perform this activity as an experiment and actually collect the data. A Data Collector (EA -100 or CBL) also could be used to gather data.

Sample resources

Mathematics SOL Curriculum Framework
http://www.pen.k12.va.us/VDOE/Instruction/Math/math_framework.html

SOL Test Blueprints
Released SOL Test Items http://www.pen.k12.va.us/VDOE/Assessment/Release2003/index.html
Virginia Algebra Resource Center http://curry.edschool.virginia.edu/k12/algebra
NASA http://spacelink.nasa.gov/.index.html
The Math Forum http://forum.swarthmore.edu/
4teachers http://www.4teachers.org
Appalachia Educational Laboratory (AEL) http://www.ael.org/pnp/index.htm
Eisenhower National Clearinghouse http://www.enc.org/

Infinite Geometric Sequences

With each bounce, ball1 reaches a height equal to 3/4 the height of its previous bounce. On the first bounce it achieves a height of 25 feet. Ball 2, which reaches a height of 18 feet on its first bounce, bounces 4/5 of its previous height.

1. Construct a table of values that shows the height of each ball for each bounce.

Bounce	Ball 1	Ball 2
1		
2		
3		
4		
5		
6		
7		

2. Will the second ball ever bounce higher than the first one? If so, at which bounce?
3. For how many bounces do both balls stay above 10 feet?
4. When do the balls "stop" bouncing (achieve a height less than 3 inches)?
5. What minimum initial height (to the nearest foot) would you have to ensure for each ball in order to guarantee that they each stay at least 8 feet above the ground by the sixth bounce?

Follow-up/extension

- What about the fractions involved in this problem make the ball react in the manner described?

Homework

The great Greek philosopher Zeon of Elea (born sometime between 495 and 480 BC) proposed four paradoxes in an effort to challenge the accepted notions of space and time that he encountered in various philosophical circles. His paradoxes confounded mathematicians for centuries, and it was not until Cantor's development of the theory of infinite sets (in the 1860s and 1870s) that the paradoxes could be fully resolved.

Zeno's paradoxes focus on the relation of the discrete to the continuous, an issue that is at the very heart of mathematics. Zeno's first paradox attacks the notion held by many philosophers of his day that space was infinitely divisible and that motion was therefore continuous.

The Motionless Runner

A runner wants to run a certain distance — say, 100 meters — in a finite time. But to reach the 100-meter mark, the runner must first reach the 50-meter mark, and to reach that, the runner must first run 25 meters. But to do that, he must first run 12.5 meters. Since space is infinitely divisible, we can repeat these "requirements" forever. Thus the runner has to reach an infinite number of midpoints in a finite time. This is impossible, so the runner can never reach his goal. In general, anyone who wants to move from one point to another must meet these requirements, and so motion is impossible, and what we perceive as motion is merely an illusion.

- Where does this argument break down?
- Why?

Sample assessment

Driving a piling into a harbor bottom, a pile driver sinks the piling 24 inches on the first stroke, 18 inches on the second stroke, and $13\frac{1}{2}$ inches on the third stroke. If the sequence is continued, how far will the piling be driven down on the fifth stroke?

F $1\frac{1}{2}$ in.

G $4\frac{1}{2}$ in.

H 6 in.

J $7\frac{19}{32}$ in.

Two arithmetic means between 3 and 24 are

A 8 and 12
B 8 and 16
C 9 and 16
D 10 and 17

Which of the following sets represents an arithmetic sequence?

A {2, 11, 20, 29, 38, ...}
B {1, 3, 9, 27, 81, ...}
C {3, –5, 7, –9, 11, ...}
D {1, 16, 36, 64, 100, ...}

If $a^n = 6 + (n - 1)5$, then $a_7 =$

F 31
G 36
H 40
J 42