Organizing Topic: Exponential Modeling

Mathematical Goals:

- Students will model exponential relationships from data gathered during activities and from Internet database sources.
- Students will investigate and analyze key characteristics of exponential functions including domain, range, asymptotes, increasing/decreasing behavior, and end behavior.
- Students will make predictions using exponential curve-fitting and evaluating the model at specific domain values outside the given data set.

Standards Addressed: AFDA.1; AFDA.2; AFDA.3; AFDA.4

Data Used:

- Data obtained from observation/measurement in activities
- Data imported from Internet databases

Materials:

- Applications: **EasyData™** and **Transformation™** Application
- Graphing calculator and links
- Handout – Who Wants to be a Millionaire?
- Handout – Paper Folding
- Handout – M&M™ Decay
- M&M™'s Fun Size, cups, paper towels
- Handout – Decaying Dice Game
- Handout – Population Growth
- Handout – Baseball Players' Salaries
- Graph paper

Instructional Activities

I. Who Wants to be a Millionaire?

Students will investigate the similarities and differences between and among constant, linear, and exponential functions. Students will gain familiarity with the graphing calculator (how to use the **Stat** functions to put data in lists, find regression equations, and use **StatPlot, ZoomStat**, **Table**, and **Tblset functions**.).

Concepts covered include:

- scatter plots;
- domain and range;
- continuity;
- linear and exponential functions;
- evaluating a function for a given domain element;
- independent and dependent variables;
- slope;
- slope-intercept form of an equation;
- linear and exponential regression; and
- transformations.

II. Paper Folding Activity

Students will model exponential growth and exponential decay functions by folding paper. Students will investigate how quickly an exponential function increases/decreases.

Concepts covered include:

- scatter plots;
- domain and range;
- continuity;
- linear and exponential functions;
- evaluating a function for a given domain element;
- independent and dependent variables;
- slope;
- slope-intercept form of an equation;
- exponential regression;
- transformations;
- function;
- exponential growth/decay;
- asymptotes; and
- end behavior.

III. M&M™ Decay

Students will collect experimental data from trials. Data will vary. An exponential decay model will best represent the data, with a rate of decay close to 0.5.

Concepts covered include:

- scatter plots;
- domain and range;
- continuity;
- linear and exponential functions;
- evaluating a function for a given domain element;
- independent and dependent variables;
- slope;
- slope-intercept form of an equation;
- exponential regression;
- transformations;
- function;
- exponential decay;
- asymptotes;
- end behavior; and
- theoretical and experimental probability.

IV. Decaying Dice Game (optional)

Students will play a game to reinforce the concepts of exponential decay and probability of events.

Concepts covered include:

- domain and range;
- continuity;
- independent and dependent variables;
- transformations;
- function;
- exponential decay;
- end behavior; and
- theoretical and experimental probability.

V. Population Growth

Students will use the Internet to collect data on average salaries of baseball players from 1975 to present. Students will analyze data and model with an appropriate function.

Concepts covered include:

- scatter plots;
- domain and range;
- continuity;
- linear and exponential functions;
- evaluating a function for a given domain element;
- independent and dependent variables;
- exponential regression;
- transformations;
- function;
- asymptotes; and
- end behavior.

VI. Baseball Players Salaries

Students will use the Internet to collect data on average salaries of baseball players from 1975 to present. Students will analyze data and model with an appropriate function.

Concepts covered include:

- scatter plots;
- domain and range;
- continuity;
- linear and exponential functions;
- evaluating a function for a given domain element;
- independent and dependent variables;
- exponential regression;
- transformations;
- function;
- asymptotes; and
- end behavior.

Activity I: Teacher Notes--Who Wants to Be a Millionaire?

Students should recognize that Options 1 and 2 are both linear and that Option 3 is not linear. Intuitively, students may feel that either Option 1 or 2 is the best until they realize how fast the value of the pennies is increasing additively, or better yet, increasing multiplicatively. This warm-up takes students from a function with which they are very familiar and compares it with the exponential function.

Have students plot the data, (days, salary) and discuss what the data look like (linear for Options 1 and 2, exponential for 3). Discuss whether the data are discrete or continuous; whether or not the relation is a function; finding the domain and range; and identifying the independent and dependent variables. Have students enter the data in the graphing calculator and determine the curve of best fit using the **Transformation™** Application or **LinReg/ExpReg™**. Discuss the meaning of the y-intercepts and the rate of change. Guide students in a discussion of the differences between linear functions and exponential functions. Have students identify key characteristics of exponential functions and define the function of *a* and *b* in $y=a(b)^x$.

Who Wants to be a Millionaire?

You are sitting in mathematics class, and the famous billionaire, Bill Buffett Jobs, offers you the job of a lifetime. You would only need to work for one month (30 days) and could become a millionaire. But there is a catch! He offers you three payment options and to show yourself worthy, you must pick the best option and explain your choice.

Option 1: You earn $1,000,000, evenly distributed over the 30 day period.

Option 2: You earn $3,000 the first day, then for each following day an additional $3,000 will be added to the previous day's salary for the 30 days.

Option 3: You earn one cent the first day, two cents the second, and double your salary each day thereafter for 30 days.

Collecting Data

1. Which option should you choose? Explain your reasoning:

Complete the following tables for each of the 3 options.

Option 1	
Day	**Salary**
1	
2	
3	
4	
5	
6	
7	

Option 2	
Day	**Salary**
1	
2	
3	
4	
5	
6	
7	

Option 3	
Day	**Salary**
1	
2	
3	
4	
5	
6	
7	

We will now enter each set of data into the calculator to determine the mathematical model for each and decide which payment option is wisest.

2. Option 1 and Option 2 can both be modeled by a linear regression equation ***y = mx+b***.

3. Option 1 can be modeled by the equation: y = ________________
Give an explanation for the slope:

4. Option 2 can be modeled by the equation: y = ________________
Give an explanation for the slope:

5. Does option 3 seem to follow a linear model? Explain your reasoning.

6. For Option 3, find an equation for the curve using regression (use **ExpReg** under **Stat:Calc** menu).

7. An equation for an exponential curve is of the form $y = a(b)^x$. Option 3 can be modeled by the equation: y = _______________ What is the significance of ***b*** in this equation?

8. Graph the three equations representing the three options simultaneously in y_1, y_2, y_3. Press the **Window** button and type in these settings to properly view all three graphs **Xmin** = 0, **Xmax** = 30, **Xscl** = 1, **Ymin** = 0, **Ymax** = 1,000,000, **Yscl**=1
What observations can you make about the three graphs?

   What do you notice about the Option 3 graph?

9. Using the **Table** function, evaluate how much your salary would be on Day 30 using Option 2 and then Option 3.

10. If you were to find the sum of all the payments under Option 3, it would be over $10,000,000! That's a lot of pennies!

Fitting the Equation

11. Using Option 3, enter the data into **[Stat] [Edit]** and perform the appropriate regression to determine the equation of the curve of best fit. Activate **Stat Plot 1** for the scatter plot (time, growth) graph. Activate **Stat Plot 2** for the scatter plot (time, **RESID**) graph. Complete the table below.

TIME L1	ACTUAL DISTANCE L2	FITTED DISTANCE L3	ACTUAL – FITTED L4 = L2 – L3	(ACTUAL – FITTED)^2 L5 = (L4)^2
0				
1				
2				
3				
4				
5				
6				
7				
8				
9				
Total Actual – Fitted [Sum(L4)]				
Total (Actual – Fitted)^2 [Sum(L5)]				

12. Use your calculator and determine another type of regression for the given data and repeat the above process recording the values in the table below.

TIME L1	ACTUAL DISTANCE L2	FITTED DISTANCE L3	ACTUAL – FITTED L4 = L2 – L3	(ACTUAL – FITTED)^2 L5 = (L4)^2
0				
1				
2				
3				
4				
5				
6				
7				
8				
9				
Total Actual – Fitted [Sum(L4)]				
Total (Actual – Fitted)^2 [Sum(L5)]				

13. Compare and contrast the data in the two tables.

Activity II. Teacher's Notes: Paper Folding Activity

The instructor may choose to introduce this activity by choosing a student and challenging him/her, "How many times do you think you can fold this sheet of paper in half?"

Have the student demonstrate how difficult the task becomes after only 6 folds.

This may be done individually or in small groups. Students will follow directions on worksheet and complete.

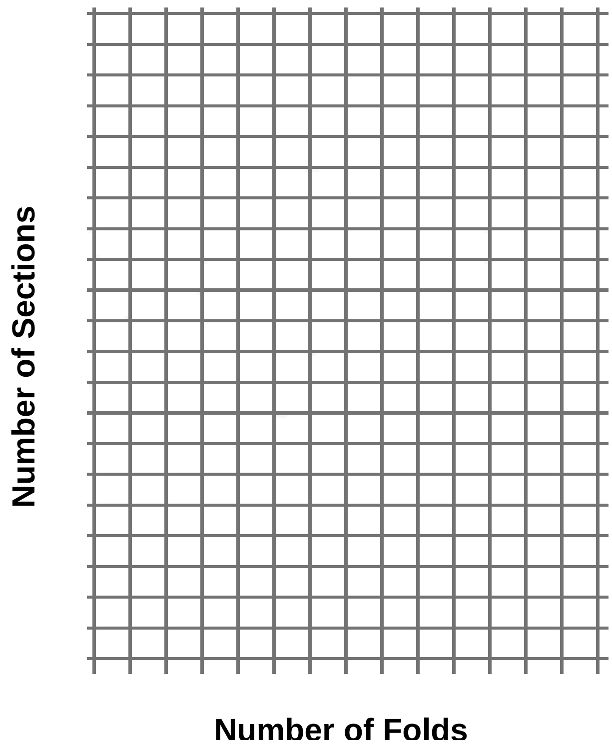
Activity II is a very hands-on, concrete example of both exponential growth and exponential decay and makes it clear what the values of ***a*** and ***b*** in the exponential equation mean in terms of their effects on the equation and in the context of the problem.

Paper Folding Activity

Number of Sections: How many times do you think you can fold a piece of paper in half?

1. Fold an 8.5" x 11" sheet of paper in half and determine the number of sections the paper has after each fold.
2. Record your data in the table below and continue folding in half until it becomes too hard to fold the paper.
3. Then make a scatter plot of your data.

Number of Sections	
Number of Folds	**Number of Sections**
0	
1	
2	
3	
4	
5	
6	
7	
8	



4. Using your calculator, determine the mathematical model that represents this data:
   y = ____________

5. Explain in words what the mathematical model means.

6. What might be different if you tried this experiment with wax paper or tissue paper?

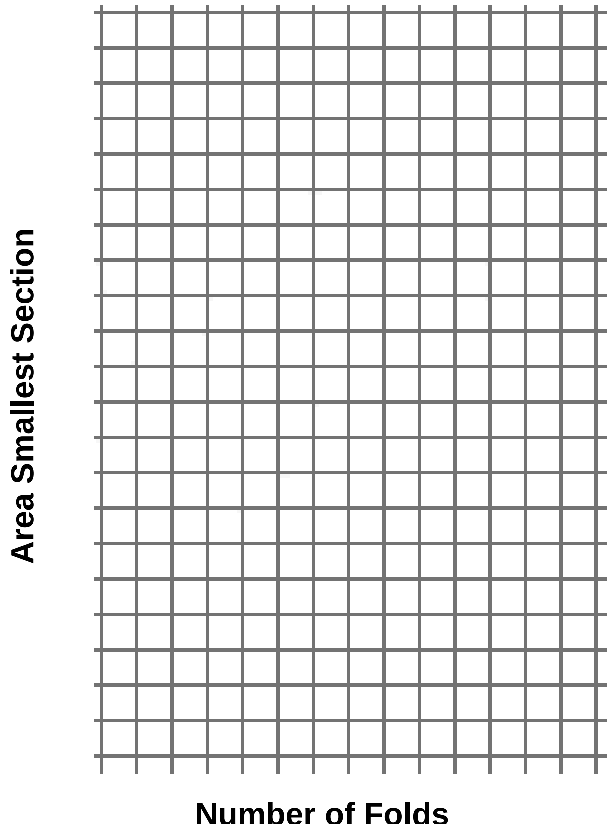
This is an example of exponential growth. The thickness of the paper grows very rapidly with each fold. To get an idea of this incredible growth, consider:

At 7 folds, it is as thick as a notebook.
At 17 folds, it would be taller than the average house.
At 20 folds, the sheet of paper is thick enough to extend a quarter of the way up the Sears Tower in Chicago.
At 30 folds, it has crossed the outer limits of the atmosphere.

Area of Smallest Section

7. Again, fold a piece of paper in half and determine the area of the smallest section after you have made a fold. What is the original area of the sheet of paper?
8. Record your data in the table below.
9. Then make a scatter plot of your data.

Number of Sections	
Number of Folds	Area of Smallest Section
0	
1	
2	
3	
4	
5	
6	
7	
8	



10. Using your calculator, determine the mathematical model that represents this data:

    y = ______________.

    This is an example of exponential decay.

11. Explain what each part of the mathematical model means.

12. What would be the area of the smallest section of the piece of paper, if you were able to fold it 10 times?

Fitting the Equation

13. Enter the data into **[Stat] [Edit]** and perform the appropriate regression to determine the equation of the curve of best fit.
Activate **Stat Plot 1** for the scatter plot (time, growth/decay) graph.
Activate **Stat Plot 2** for the scatter plot (time, **RESID)** graph.
Complete the table below.

TIME L1	ACTUAL DISTANCE L2	FITTED DISTANCE L3	ACTUAL – FITTED L4 = L2 – L3	(ACTUAL – FITTED)^2 L5 = (L4)^2
0				
1				
2				
3				
4				
5				
6				
7				
8				
9				
Total Actual – Fitted [Sum(L4)]				
Total (Actual – Fitted)^2 [Sum(L5)]				

14. Use your calculator and determine another type of regression for the given data and repeat the above process recording the values in the table below.

TIME L1	ACTUAL DISTANCE L2	FITTED DISTANCE L3	ACTUAL – FITTED L4 = L2 – L3	(ACTUAL – FITTED)^2 L5 = (L4)^2
0				
1				
2				
3				
4				
5				
6				
7				
8				
9				
Total Actual – Fitted [Sum(L4)]				
Total (Actual – Fitted)^2 [Sum(L5)]				

15. Compare and contrast the data in the two tables.

Activity III: Teacher Notes--M&M™ Decay

M&M™ Decay is a group activity (3-4 students in a group). Students will perform the experiment and collect their own data from a series of trials.

Students will use a cup to mix M&M™'s in a variety of colors and place them on a paper towel on the desk. It is important that students do not use a value of 0 for ***Number of M&M'™s Remaining*** so that they can perform an **ExpReg™** on their data.

This activity provides a concrete example of exponential decay. Through this activity, students should begin to understand how exponential growth and exponential decay can occur as natural phenomena.

Extension is optional but provides further investigation into exponential functions.

M&M™ DECAY

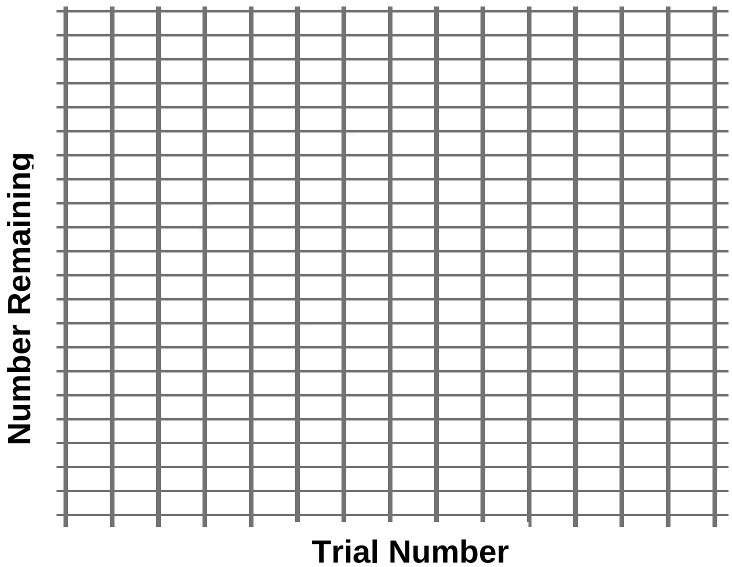
Collecting Data

Empty your bag of M&M™'s onto the table and count the M&M™'s. Then place the M&M™'s in a cup and mix them well. Pour them out on the desk, count the number that show an "m," and place them back in the cup. The others may be eaten or removed. Record the number of M&M™'s that show an "m" in your data table. Then repeat the procedure. Continue until the number of M&M™'s remaining is less than 5, but greater than 0.

Number of M&M's	
Trial Number	**Number of M&M's Remaining**
0 (initial amount)	
1	
2	
3	
4	
5	
6	
7	
8	

Graphing and Determining the Model

Use a graphing calculator to make a scatter plot of your data. Copy your scatter plot onto the grid below. Then use the graphing calculator to find the curve of best fit, and graph the equation. Sketch in the curve and write your equation.



Equation:________________

Interpreting the Data

1. In your model, ***y = a(b)***^{x}, what value do you have for ***a***? To what does ***a*** seem to relate when you consider your data? When x = 0, what is your function value? Compare this to the values in your data table.

2. What is the theoretical probability that any single M&M™ will be removed in a trial?

3. What is the value for ***b*** in your exponential model? Explain the significance of this value and how it relates to your data.

4. If you started with 40 M&M™'s, how many trials do you think it would take before the number of M&M™'s was between five and zero? What equation would model this new, initial value?

5. How does the M&M™ experiment compare with the paper-folding activity? How are they alike and how are they different?

Extension—Beyond M&M™ Decay

6. What other objects could be used that would follow the same exponential model as in the previous experiment? What objects could you use to change the value of ***b***?

7. How could you use M&M™'s to model exponential growth instead of exponential decay?

8. Use the Internet to research real-life phenomena that follow either an exponential growth model or an exponential decay model.

Fitting the Equation

9. Enter the data above that illustrates exponential decay in to **[Stat] [edit]** and perform the appropriate regression to determine the curve of best fit. Activate **Stat Plot 1** for the scatter plot (time, decay) graph. Activate **Stat Plot 2** for the scatter plot (time, RESID) graph. Complete the table below.

TIME L1	ACTUAL DISTANCE L2	FITTED DISTANCE L3	ACTUAL – FITTED L4 = L2 – L3	(ACTUAL – FITTED)^2 L5 = (L4)^2
0				
1				
2				
3				
4				
5				
6				
7				
8				
9				
Total Actual – Fitted [Sum(L4)]				
Total (Actual – Fitted)^2 [Sum(L5)]				

10. Use your calculator and determine another type of regression for the given data and repeat the above process recording the values in the table below.

TIME L1	ACTUAL DISTANCE L2	FITTED DISTANCE L3	ACTUAL – FITTED L4 = L2 – L3	$(\text{ACTUAL – FITTED})^2$ L5 = $(L4)^2$
0				
1				
2				
3				
4				
5				
6				
7				
8				
9				
Total Actual – Fitted [Sum(L4)]				
Total $(\text{Actual – Fitted})^2$ [Sum(L5)]				

11. Compare and contrast the data in the two tables.

Activity IV: Teacher Notes--Decaying Dice Game (optional)

This is a group activity (2 to 4 students per group). Students may play the game in a small group to reinforce the concepts in the **M&M™ Decay** exploration. This activity can be used if some groups finish the **M&M™ Decay** activity early. Students will have to calculate probabilities, and make predictions, based on a new exponential decay model.

Decaying Dice Game

Objective of the game: To predict as closely as possible how long your dice will "stay alive".

Instructions: Before a player rolls the dice, the player makes a prediction of how many throws of the dice they will have until all dice are removed. Dice that come up "6" are removed, and the remaining dice are thrown. Repeat the process and keep count of how many rolls it takes to remove all the dice. In the subsequent rounds, remove one die at the beginning of the game.

Scoring: 50 points, if the number of actual rolls matches the player's prediction
25 points, if it is 1 number above or below the prediction
10 points, if it is 2 away from the prediction.

Alternatives: Choose multiple numbers to be removed or choose a number that "reproduces" another die to be added.

Player

	Prediction	Actual	Score
Round 1			
Round 2			
Round 3			
Round 4			
Round 5			
Round 6			
Total Score			

Player

	Prediction	Actual	Score
Round 1			
Round 2			
Round 3			
Round 4			
Round 5			
Round 6			
Total Score			

Player

	Prediction	Actual	Score
Round 1			
Round 2			
Round 3			
Round 4			
Round 5			
Round 6			
Total Score			

Player

	Prediction	Actual	Score
Round 1			
Round 2			
Round 3			
Round 4			
Round 5			
Round 6			
Total Score			

Activity V: Teacher Notes--Population Growth

Population Growth can be used as an individual or group activity and will require Internet access. Students will collect data from **U.S. Census Bureau International Database** to compare population growth in three countries. We suggest that students use Ethiopia, U.S., and China as the three countries and these three display very different characteristics. Ethiopia seems to follow a strong exponential model, the U.S. appears very linear, and China can best be described as logistic or S-shaped. If other countries are chosen, we suggest that the countries be located in very different geographical regions.

The instructor might choose to have students complete part or all of this activity outside of the classroom as a special project. The instructor might also choose to expand upon this activity and collaborate with a history/social studies teacher and ask students to go into more depth by describing the similarities and differences of the countries and the factors that may influence human population growth.

This activity might also be modified to look at animal populations of three different species in a specific region.

Population Growth

Using the Internet to access the U.S. Census Bureau's International Database, you will collect population data from three countries. You will analyze the data and decide which type of model best represents the data.
The Internet site: http://www.census.gov/ipc/www/idb/

Collect the Data

1. From this site, go to the "Tables" link. We will be looking at total midyear population. Go to the bottom of the screen and hit "Submit Query." On the next screen, scroll down to the list of countries. Holding down the Control key, select China, Ethiopia, and the United States (other countries may be used with teacher approval). Scroll further down to the Year Selection and type in From: 1950 To: 2005 By: 5. Then select Go. The next screen should give you a table of populations for the selected countries.

Country:		**Country:**		**Country:**	
Year	**Population**	**Year**	**Population**	**Year**	**Population**
1950		**1950**		**1950**	
1955		**1955**		**1955**	
1960		**1960**		**1960**	
1965		**1965**		**1965**	
1970		**1970**		**1970**	
1975		**1975**		**1975**	
1980		**1980**		**1980**	
1985		**1985**		**1985**	
1990		**1990**		**1990**	
1995		**1995**		**1995**	
2000		**2000**		**2000**	
2005		**2005**		**2005**	

Graph and Determine the Model

2. With a graphing utility, create a scatter plot by entering data in the tables above into your calculator (**L1** = year, **L2** = population of country 1, **L3** = population of country 2, and **L4** = population of country 3). Examine the data and determine what mathematical model you think best represents the data (it may be linear, exponential or perhaps something else!) Draw your graphs on a separate piece of graph paper.

Country 1:________________Regression Equation: y = ____________________

Country 2:________________Regression Equation: y = ____________________

Country .3:_______________ Regression Equation: y = ____________________

Interpret the Data

3. Compare and contrast your observations of the scatter plots and graphs.

4. Discuss the domain, range, shape, and end behavior of each.

5. Which country has the fastest growth rate? Which has the smallest growth rate?

6. Based on your models, make a prediction for the future populations of each country in ten years.

7. Compare your predictions to the predictions of the U.S. Census Bureau by revisiting the Web site and selecting the countries and designated year. How do they compare?

Extension

8. Discuss several factors that influence human population growth.

Want to see how the world's population is growing right now?
Go to: http://www.worldometers.info/

Fitting the Equation

9. Select one of the three countries and use the regression model that applies to the population data for that country and was found in Part 2. Complete the table below. In the column representing year, let 0 represent 1950; represent 1951, and so on.

TIME L1	ACTUAL DISTANCE L2	FITTED DISTANCE L3	ACTUAL – FITTED L4 = L2 – L3	(ACTUAL – FITTED)^2 L5 = (L4)^2
0				
1				
2				
3				
4				
5				
6				
7				
8				
9				
Total Actual – Fitted [Sum(L4)]				
Total (Actual – Fitted)^2 [Sum(L5)]				

Enter the data above into **L1, L2, and L3**.
Activate **Stat Plot 1** for the scatter plot (time, population) (**L1, L2**) graph.
Activate **Stat Plot 2** for the scatter plot (time, residual) (**L3, L4**) graph.

10. Select one of your other choices and determine the regression for the given data. Repeat the above process recording the values in the table below.

TIME L1	ACTUAL DISTANCE L2	FITTED DISTANCE L3	ACTUAL – FITTED L4 = L2 – L3	(ACTUAL – FITTED)^2 L5 = (L4)^2
0				
1				
2				
3				
4				
5				
6				
7				
8				
9				
Total Actual – Fitted [Sum(L4)]				
Total (Actual – Fitted)^2 [Sum(L5)]				

11. Compare and contrast the data in the two tables.

Activity VI: Teacher Notes--Baseball Players' Salaries

Baseball Players' Salaries is an individual activity and requires Internet access for each participant. Students must complete the handout. We suggest that students do their own search.

One Web site that works: http://eh.net/encyclopedia/article/haupert.mlb.

Baseball Players' Salaries

Collecting Data

1. Use the Internet to find the average salaries of major league baseball players for selected years 1975 to the present. Write the URL you used below. Put your data into tabular form.

2. Using a graphing utility, draw a scatter plot for this data comparing the average salaries of major league players over the years.

3. Find the equation that best fits the data: y = ______________

4. On graph paper, draw the scatter plot and regression curve.

5. What is the correlation coefficient for this data? (Use the **DiagonisticOn** function on the calculator and the correlation coefficient is ***r***.)

6. Extrapolate from the data (make a prediction) what the average baseball player's salary will be in 5 years; in 10 years.

7. Does this model seem reasonable to you? Why do you think baseball players' salaries have followed this model? Do you think other occupations' salaries would follow a similar model?

Fitting the Equation

8. Enter the baseball salary data in to **[Stat] [Edit]** and perform the appropriate regression to determine the curve of best fit. Activate **Stat Plot 1** for the scatter plot (time, salary) graph. Activate **Stat Plot 2** for the scatter plot (time, **RESID)** graph. Complete the table below.

TIME L1	ACTUAL DISTANCE L2	FITTED DISTANCE L3	ACTUAL – FITTED L4 = L2 – L3	(ACTUAL – FITTED)^2 L5 = (L4)^2
0				
1				
2				
3				
4				
5				
6				
7				
8				
9				
Total Actual – Fitted [Sum(L4)]				
Total (Actual – Fitted)^2 [Sum(L5)]				

9. Use your calculator and determine another type of regression for the given data. Repeat the above process recording the values in the table below.

TIME L1	ACTUAL DISTANCE L2	FITTED DISTANCE L3	ACTUAL – FITTED L4 = L2 – L3	(ACTUAL – FITTED)^2 L5 = (L4)^2
0				
1				
2				
3				
4				
5				
6				
7				
8				
9				
Total Actual – Fitted [Sum(L4)]				
Total (Actual – Fitted)^2 [Sum(L5)]				

10. Compare and contrast the data in the two tables.

Other Topics for Exploration in Exponential Growth and Decay

- Calculating compound interest
- Population growth of species
- Limitations of exponential models and logistic curves
- Exponential stories: Rice on a chessboard; Waterlilies
- Cell growth in biology
- Radioactive decay
- Processing power of computers and Moore's Law

Resources:

The Math Forum – Exploring Data: Exploring Data, Courses and Software
http://mathforum.org/workshops/usi/dataproject/usi.genwebsites.html

- The Data Library – Pat Daley

  This site includes collaborative projects - specific data collection projects that teachers and their students may become a part of; data sets that can be downloaded then sorted, manipulated, and graphed; and other sources of data - sites like the Bureau of Labor Statistics and the Chance Database that offer many more data sets in other formats.

Math Tools – Math Tools is a project of The Math Forum @ Drexel, funded in part by the National Science Foundation.
http://mathforum.org/mathtools/sitemap2/a2/

- Math Topics for Algebra II

U.S. Census Bureau – International Program
http://www.census.gov/ipc/www/idb/

- International Database

Economic History Association
http://eh.net/encyclopedia/article/haupert.mlb

- Scroll down to Average Salaries of Baseball Players

Real Time World Statistics - Worldometers
http://www.worldometers.info/

- World statistics updated in real time