Rational Equations

Reporting Category Equations and Inequalities

Topic Solving equations containing rational algebraic expressions

Primary SOL AII.4c The student will solve, algebraically and graphically, equations containing rational algebraic expressions. Graphing calculators will be used for solving and for confirming the algebraic solutions.

Related SOL AII.1a, d

Materials
- Graphing calculators
- Four attached handouts

Vocabulary

factor, solution, replacement set, algorithm, numerator, denominator, least common multiple (earlier grades)

rational algebraic equations, extraneous solution (AII.4.c)

Student/Teacher Actions (what students and teachers should be doing to facilitate learning)

1. Distribute copies of the attached Solving Rational Equations: Introductory Exercise handout, and have students complete it, working individually and then in pairs to share and confirm or revise their responses. Follow with a class discussion of each problem.
2. To explore the algebraic and graphical methods for solving rational expressions, begin with the algebraic. Distribute copies of the attached Steps for Solving Rational Equations Algebraically handout. Encourage students to work with their partners to monitor and communicate what is happening as you lead them through the examples. After you work through each example, have a student pair come up and work the similar, accompanying problem. The variety of problems is meant to encompass the scope of typical Algebra II problems.
3. Distribute copies of the attached Steps for Solving Rational Equations Graphically handout, and lead students through the steps in each example, using graphing calculators.
4. Define *extraneous solution* as a result that is not a solution to an equation even though it was obtained by correctly using an equation-solving algorithm. In rational equations, extraneous solutions always result in a division by zero error.

Assessment
- **Questions**
  - How do you solve a rational equation algebraically? Explain.
  - Given a rational algebraic equation, if you graph one side as Y_1 and the other as Y_2, how do to determine the solution set to the original equation? What do you notice about the graphs?

- **Journal/Writing Prompts**
  - Explain how you can use a graph’s points of intersection to solve a rational equation.
  - In your own words, explain what is meant by the term *extraneous solution.* Is it a solution or not? Explain why.
- **Other**
  - Give students solution sets, and ask them to create matching rational equations. (Note: Such open-ended problems allow students to be creative and differentiate the task based upon their own level of understanding.)

Extensions and Connections (for all students)

- Guide students to make connections to graphing rational functions, paying particular attention to restrictions on the domain and vertical asymptotes.
- Have students complete the attached Solving Rational Equations: Practice Problems.
- Introduce equations with complex algebraic fractions.

Strategies for Differentiation

- Have students solve rational equations by expressing each side as a single fraction first.

Solving Rational Equations: Introductory Exercise

Determine which of the following are true, and justify your answers.

1. $x=2$ is a solution to $\dfrac{3x+6}{x-1}=\dfrac{x^2+8}{x^2-3}$.

2. $x=-3$ is a solution to $\dfrac{-2x+1}{1-x}=\dfrac{3x-2}{4}$.

3. $x=4$ is a solution to $\dfrac{5x-4}{2x-8}=\dfrac{20}{x^2-16}$.

4. The solution set for the equation, $\dfrac{2}{x-3}+\dfrac{5}{x-2}=\dfrac{7x-19}{x^2-5x+6}$, is all real numbers except 3 and 2.

5. The equation, $\dfrac{x^2-10}{x+1}=\dfrac{-3x}{x+1}$, has exactly 2 solutions. (Hint: Determine the answer by using your graphing calculator to graph $y=\dfrac{x^2-10}{x+1}$ and $y=\dfrac{-3x}{x+1}$.)

Steps for Solving Rational Equations Algebraically

Example 1: $\frac{2x}{3}+\frac{x-2}{5}=\frac{1}{6}$

Step 1: Multiply both sides of the equation by the least common multiple of the denominators. In this case, the LCM of 3, 5, and 6 is 30.

$$30\times\left(\frac{2x}{3}+\frac{x-2}{5}\right)=\left(\frac{1}{6}\right)\times 30 \quad \Rightarrow 30\left(\frac{2x}{3}\right)+30\left(\frac{x-2}{5}\right)=5$$

Step 2: Simplify and solve familiar equation.

$$\Rightarrow\ 20x+6(x-2)=5$$
$$\Rightarrow\ 26x-12=5$$
$$\Rightarrow\ 26x=17$$
$$\Rightarrow\ x=\frac{17}{26}$$

Step 3: Verify solution.

Does $\frac{2\left(\frac{17}{26}\right)}{3}+\frac{\left(\frac{17}{26}\right)-2}{5}=\frac{1}{6}$?

Problem 1: Now, you follow the steps to solve $\frac{x+1}{2}-\frac{5x}{3}=-x$.

Example 2: $\frac{4}{x+2}+\frac{5}{x-2}=\frac{29}{x^2-4}$

Step 1: Multiply both sides of the equation by the least common multiple of the denominators. In this case, the LCM of $x + 2$, $x - 2$, and $x^2 - 4$ is $(x + 2)(x - 2)$.

$$(x+2)(x-2)\left(\frac{4}{x+2}+\frac{5}{x-2}\right)=\left(\frac{29}{x^2-4}\right)(x+2)(x-2)$$

$$\Rightarrow (x+2)(x-2)\left(\frac{4}{x+2}\right)+(x+2)(x-2)\left(\frac{5}{x-2}\right)=\left(\frac{29}{x^2-4}\right)(x+2)(x-2)$$

Step 2: Simplify and solve familiar equation.

$$\Rightarrow \cancel{(x+2)}(x-2)\left(\frac{4}{\cancel{x+2}}\right)+(x+2)\cancel{(x-2)}\left(\frac{5}{\cancel{x-2}}\right)=\left(\frac{29}{\cancel{x^2-4}}\right)\cancel{(x+2)(x-2)}$$

$\Rightarrow\ 4x-8+5x+10=29$

$\Rightarrow\ 9x+2=29$

$\Rightarrow\ 9x=27$

$\Rightarrow\ x=3$

Step 3: Verify solution.

Does $\frac{4}{x+2}+\frac{5}{x-2}=\frac{29}{x^2-4}$ when $x=3$?

Problem 2: Now, you follow the steps to solve $\frac{7}{x+2}-\frac{4}{x-3}=\frac{-14}{x^2-x-6}$.

Example 3: $\frac{x}{x-1}+\frac{3}{x}=\frac{5}{2x}$

Step 1: Multiply both sides of the equation by the least common multiple of the denominators. In this case the LCM of $x-1$, x, and $2x$ is $2x(x-1)$.

$$\Rightarrow 2x\cancel{(x-1)}\left(\frac{x}{\cancel{x-1}}\right)+2\cancel{x}(x-1)\left(\frac{3}{\cancel{x}}\right)=\left(\frac{5}{\cancel{2x}}\right)\cancel{2x}(x-1)$$

Step 2: Simplify and solve familiar equation.

$$\Rightarrow 2x^2+6x-6=5x-5$$
$$\Rightarrow 2x^2+x-1=0$$
$$\Rightarrow (2x-1)(x+1)=0$$
$$\Rightarrow x=\frac{1}{2} \text{ or } x=-1$$

Step 3: Verify solution.

Does $\frac{x}{x-1}+\frac{3}{x}=\frac{5}{2x}$ when $x=\frac{1}{2}$? $x=-1$?

Problem 3: Now, you follow the steps to solve $\frac{x}{x+1}=\frac{5}{2x-2}-\frac{1}{2}$.

Example 4: $\dfrac{x+3}{x+2}=1-\dfrac{x+1}{x+2}$

Step 1: Multiply both sides of the equation by the least common multiple of the denominators. In this case the LCM is just $x + 2$.

$$\Rightarrow (x+2)\left(\frac{x+3}{x+2}\right)=\left(1-\frac{x+1}{x+2}\right)(x+2)$$

Step 2: Simplify and solve familiar equation.

$$\Rightarrow x+3=(x+2)-(x+1)$$
$$\Rightarrow x+3=1$$
$$\Rightarrow x=-2$$

Step 3: Verify solution.

When we check $x = -2$, we can see that it leads to division by zero. Because we correctly followed the process for finding a solution and the result we generated does not solve the equation, it is call an extraneous solution. Thus, the answer to the problem is "no solutions."

Problem 4: Now, you follow the steps to solve $\dfrac{4x}{x+1}+\dfrac{x+5}{x+1}=3$.

Example 5: $\frac{x+3}{x+2}=2-\frac{x+1}{x+2}$ (note how similar this is to Example 4.)

Step 1: Multiply both sides of the equation by the least common multiple of the denominators. In this case, the LCM is just $x+2$.

$\Rightarrow (x+2)\left(\frac{x+3}{x+2}\right)=\left(2-\frac{x+1}{x+2}\right)(x+2)$

Step 2: Simplify and solve familiar equation.

$\Rightarrow x+3=(2x+4)-(x+1)$

$\Rightarrow x+3=x+3$

$\Rightarrow x$ can be any real number and satisfy $x+3=x+3$, but when we want to solve $\frac{x+3}{x+2}=2-\frac{x+1}{x+2}$, x cannot equal −2. Why?

So, the solution set is all real numbers except −2, or in set-builder notation, $\{x \mid x \neq -2\}$.

Step 3: Verify solution.

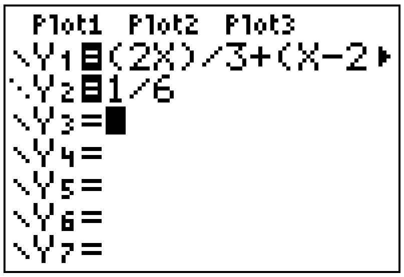
It is impossible to check all real numbers as solutions, but verify a couple.

Problem 5: Now, you follow the steps to solve $\frac{4x}{x+1}+\frac{x+5}{x+1}=5$.

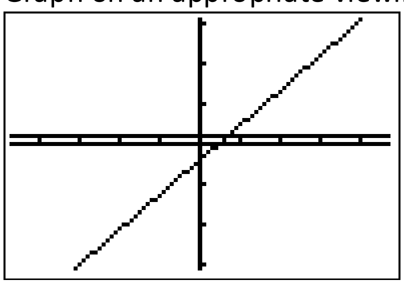
Steps for Solving Rational Equations Graphically

Example 1: $\frac{2x}{3}+\frac{x-2}{5}=\frac{1}{6}$

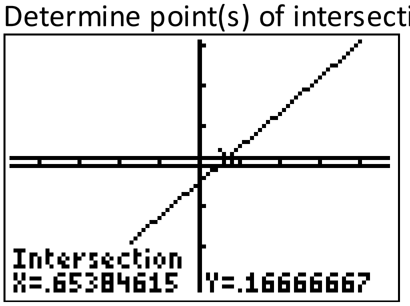
Step 1: Enter the left-hand side of the equation in Y_1 and the right-hand side in Y_2.



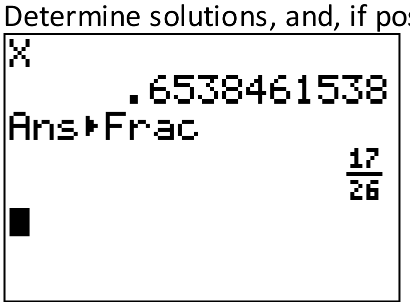
Step 2: Graph on an appropriate viewing window.



Step 3: Determine point(s) of intersection.



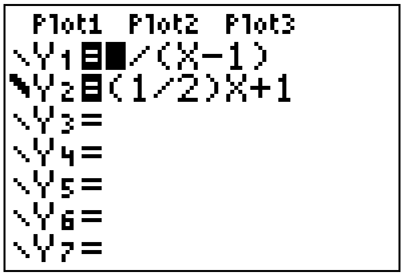
Step 4: Determine solutions, and, if possible/necessary, convert to a fraction.



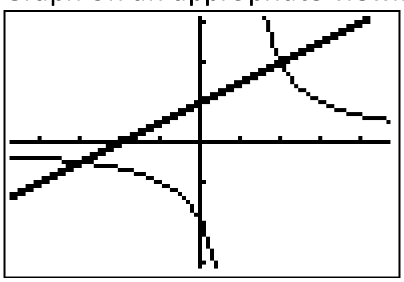
Step 5: Verify solution.

Example 2: $\frac{2}{x-1}=\frac{1}{2}x+1$

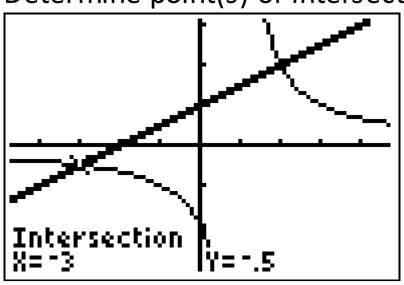
Step 1: Enter the left-hand side of the equation in Y_1 and the right-hand side in $Y_{2.}$

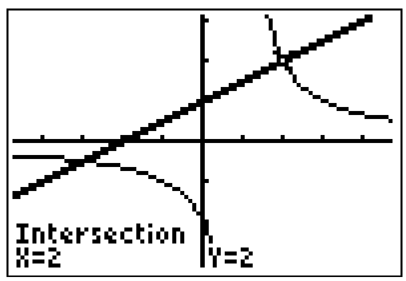


Step 2: Graph on an appropriate viewing window.



Step 3: Determine point(s) of intersection.





Step 4: Determine solutions.
The solutions are −3 and 2.

Step 5: Verify solutions.

Solving Rational Equations: Practice Problems

Solve each of the following. Be sure to check solutions in the original equations and identify any extraneous solutions.

1. $\frac{4}{x}+\frac{1}{3x}=9$
2. $\frac{3}{n+1}=\frac{5}{n-3}$
3. $\frac{2}{x+5}-\frac{3}{x-4}=\frac{6}{x}$
4. $\frac{1}{x-5}+\frac{1}{x-5}=\frac{4}{x^2-25}$
5. $\frac{6x^2+5x-11}{3x+2}=\frac{2x-5}{5}$
6. $\frac{3}{x-1}-\frac{4}{x-2}=\frac{2}{x+1}$
7. $\frac{x}{x^2-4x-12}=\frac{x+1}{6-x}-\frac{x-3}{2+x}$
8. $\frac{c+2}{c-5}=\frac{7}{c+2}$
9. $\frac{x^2-2x-3}{x^2-x-6}-\frac{x}{x+2}=\frac{5-x}{x-3}$

Solve each of the following, showing all your work:

10. A number minus 5 times its reciprocal is 7. Find the number.
11. A swimming pool can be filled in 12 hours, using the large pipe alone and in 18 hours with the small pipe alone.
    - If both pipes are used, how long will it take to fill the pool?
    - If the fire department is asked to help out by using its tank, which alone can fill the pool in 15 hours, how long will it take to fill the pool if all three water sources are used simultaneously?
    - At the end of the summer a drain is left open, and the pool empties in 24 hours. If the pool attendant who is preparing the pool for opening day forgets to plug the drain, can the pool be filled in one day? If so, how long will it take? If not, explain why not.
12. John is deciding between two jobs. One promises convenient hours and is closer to his home. The other pays $2.25 more per hour, and John would earn $1,000 in 10 hours less than it will take him to earn $900 at the first job. What is the hourly wage for each job?
13. The current in a river is estimated to be 4 mph. A speedboat goes downstream 6 miles and comes back 6 miles in 15 minutes. What is the average speed of the speedboat in still water? (Hint: Let r be the average speed in mph in still water.)
14. Find all real numbers x, $x + 2$, and $x + 4$ such that the reciprocal of the smallest number is the sum of the reciprocals of the other two. Are there rational solutions to this problem?
15. A manufacturing company has a machine that produces 120 parts per hour. When a new machine is delivered that makes twice as many parts per hour, how long will it take the two machines working together to make 120 parts per hour? (Hint: Let x be the time [number of hours] the machines take to make the 120 parts.)