Rational Functions: Intercepts, Asymptotes, and Discontinuity

Reporting Category Functions

Topic Exploring asymptotes and discontinuity

Primary SOL AII.7 The student will investigate and analyze functions algebraically and graphically. Key concepts include
- a) domain and range, including limited and discontinuous domains and ranges;
- b) zeros;
- c) *x*- and *y*-intercepts;
- e) asymptotes.

Graphing calculators will be used as a tool to assist in investigation of functions.

Related SOL AII.7d, f; AII.1d

Materials
- Graphing calculators
- Two attached handouts
- What's My Function? Graphs (attached)
- Rational Functions sheets (attached)

Vocabulary

x-intercept, y-intercept, domain, range, zeros (earlier grades)

discontinuous domain, discontinuous range, point of discontinuity, asymptote (AII.7)

Student/Teacher Actions (what students and teachers should be doing to facilitate learning)

DAY 1

1. Do a quick review of factoring, including factoring out a common factor, the difference of two squares, and a trinomial with a leading coefficient greater than one.
2. Have students graph with or without a calculator the function $f(x)=\frac{x+2}{x+3}$. Then, have them discuss their graphs with partners and write down any observations they can make about the function. Hold a class discussion with students sharing their observations.
3. Have students complete by hand a table of values for the function, using the trends of the graph to lead them to their chosen domain values. If they are not already doing so after a couple of minutes, have them choose domain values that are smaller than −50 and greater than 50, and values surrounding and including −3. Next have them see what the value of the function is when x is 100 and −100, and when x is −2.999 and −3.001.
4. Introduce the term *asymptote*. If you have already taught end behavior and increasing/decreasing intervals, talk about the behaviors surrounding the asymptote in

those terms, saying, for example, “A horizontal asymptote of a rational function represents end behavior.”

5. Have students graph $f(x)=\dfrac{x^2-16}{x-4}$ on their calculators and then simplify the rational expression on paper. Next, have them graph a second function, $y=x+4$, on their calculators without deleting the first function. Discuss why it makes sense that this second graph would be the same line. Then, discuss why it cannot be the same line. Once students have realized that *x* cannot be 4 in the original function, have them clear the second function off their calculators and take a closer look at the rational function. Students can zoom in, trace the line, and choose an *x*-value of 4, or look at a table to discover that there is actually a “hole” in the graph. Have students consider why *x* = 4 is not in the domain of *f*.
6. Introduce discontinuous functions, discontinuous domain, and point of discontinuity, explaining each.
7. Now, have students graph $f(x)=\dfrac{x-4}{x^2-16}$, factor the denominator, and discover which values of *x* will make the function undefined. Have partners write down everything they notice about this graph. Hopefully, they will discover there is a horizontal asymptote (*y* = 0) as well as a vertical asymptote (*x* = −4). Be sure they realize there is discontinuity where *x* = 4. Trace the graph with your finger, exaggerating the point of discontinuity.
8. Have student partners graph the following functions:

   a. $f(x)=\dfrac{x}{x^2}$ b. $f(x)=\dfrac{x}{x+1}$ c. $f(x)=\dfrac{3x}{x+1}$ d. $f(x)=\dfrac{x^2}{x+1}$

   Direct students to compare and contrast the functions algebraically and graphically. Challenge them to make conclusions about horizontal asymptotes based on the algebraic form of the function, as follows:
   - Degree of numerator > degree of denominator $\Rightarrow$ no horizontal asymptote
   - Degree of numerator < degree of denominator $\Rightarrow$ horizontal asymptote at *y* = 0
   - Degree of numerator = degree of denominator $\Rightarrow$ horizontal asymptote at

   $$y=\frac{\text{coefficient of hightest degree term in numerator}}{\text{coefficient of hightest degree term in denominator}}$$
9. Have students look back at all of the functions they graphed and determine the *x*- and *y*-intercepts. Stress that the zeros of the function are the *x*-intercepts and that they can often be easily found once a numerator is factored.
10. Distribute copies of the attached Investigating Functions handout, and have students work in pairs to complete #1–6 without using calculators. As two sets of partners complete the handout, have the four students compare answers and discuss any differences. Once each pair has checked and worked through the questions with another pair, have groups of four pair up to compare their work in groups of eight. Finally, have students graph the functions on their calculators to check their answers.
11. Distribute copies of the attached What’s My Function? handout. Have students play a game, competing to see who can write the most functions algebraically. Calculators may not be used! Tape the attached What’s My Function? Graphs on the board, and once

students have completed the handout; have them write their responses with their initials underneath the corresponding graphs on the board. When they have completed this task, have them graph the various student responses on their calculators to determine which functions are correct. The student who wrote the most correct functions wins a prize. (Note: Depending on your class, you may want to make this a partner or larger-group competition.)

DAY 2

1. Extend the discussion of asymptotes and intercepts to include exponential functions. Have students now complete #7–12 on the Investigating Functions handout. If you have already taught end behavior and domain and range, have students complete the Extension exercise.
2. Place the attached Rational Functions sheets across the top of the board. For each function, write "*x*-intercepts, *y*-intercepts, horizontal asymptotes, vertical asymptotes," and "points of discontinuity" on separate lines below the function. Give each student an answer card. (If there are extra cards, some students will receive more than one.) Have students place their cards in the appropriate spaces.

Assessment

- **Questions**
  - What are all asymptotes and points of discontinuity for $f(x)=\dfrac{x^2-5x-14}{2x^2-8}$?
  - What are the *x*- and *y*-intercepts of $f(x)=(-2)^x+3$? Find the answer without using a calculator.
- **Journal/Writing Prompts**
  - Describe how you can determine without graphing whether or not a rational function has any horizontal asymptotes and what the horizontal asymptotes are.
  - Explain how simplifying a rational function can help you determine any vertical asymptotes or points of discontinuity for the function.

Extensions and Connections (for all students)

- Have each student draw his/her own graph with vertical and/or horizontal asymptotes and give the graph to a classmate to write the algebraic function that is graphed.
- Discuss how transformations on the functions would affect location of asymptotes.

Strategies for Differentiation

- Have students play a matching game, either with cards or an interactive whiteboard, to match functions to their graphs.
- Use pictorial representations to reinforce vocabulary.
- Have student find and utilize a Web site that demonstrates the relationship between a function and its asymptotes, discontinuity, and intercepts.
- Make extensive use of graphing calculators and/or graphing calculator software to reinforce the characteristics of a rational function.

Investigating Functions

For each of the following functions, determine the *x*- and *y*- intercepts, discontinuous points, and vertical and horizontal asymptotes.

1. $f(x)=\dfrac{x+2}{x+3}$ *x*- and *y*-intercepts ____________ discontinuous points ____________

   vertical and horizontal asymptotes __________________

2. $f(x)=\dfrac{x^2-9}{x-3}$ *x*- and *y*-intercepts ____________ discontinuous points ____________

   vertical and horizontal asymptotes __________________

3. $f(x)=\dfrac{2x^2-5x-12}{x^2-16}$ *x*- and *y*-intercepts ____________ discontinuous points ____________

   vertical and horizontal asymptotes __________________

4. $f(x)=\dfrac{5}{x-6}$ *x*- and *y*-intercepts ____________ discontinuous points ____________

   vertical and horizontal asymptotes __________________

5. $f(x)=\dfrac{x+2}{x^2-4}$ *x*- and *y*-intercepts ____________ discontinuous points ____________

   vertical and horizontal asymptotes __________________

6. $f(x)=\dfrac{1}{x}$ *x*- and *y*-intercepts ____________ discontinuous points ____________

   vertical and horizontal asymptotes __________________

7. $y = 2^x$ x- and y-intercepts ____________ discontinuous points ____________
vertical and horizontal asymptotes __________________

8. $y = 2^{x+2}$ x- and y-intercepts ____________ discontinuous points ____________
vertical and horizontal asymptotes __________________

9. $y = -(2)^{x-2}$ x- and y-intercepts ____________ discontinuous points ____________
vertical and horizontal asymptotes __________________

10. $3xy = 6$ x- and y-intercepts ____________ discontinuous points ____________
vertical and horizontal asymptotes __________________

11. $y = \frac{3}{x}$ x- and y-intercepts ____________ discontinuous points ____________
vertical and horizontal asymptotes __________________

12. $-2xy = 1$ x- and y-intercepts ____________ discontinuous points ____________
vertical and horizontal asymptotes __________________

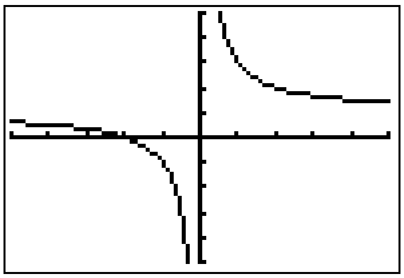
Extension

On your calculator, graph each of the 12 functions above individually. State the domain and range of each function, and describe the end behavior.

What's My Function?

Write the algebraic function for each graph shown.

1.



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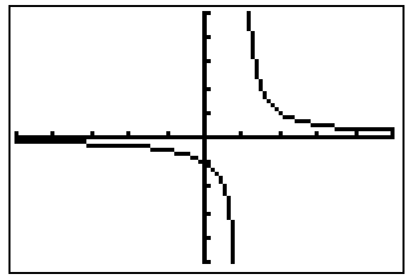
2.



$x = -4$ $y =$

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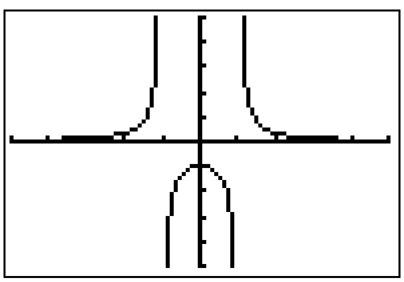
3.



$x = 0$ $y =$

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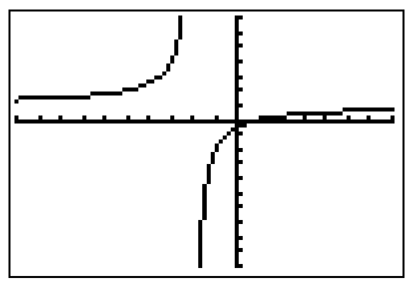
4.



$x = 1$ $y = 0$

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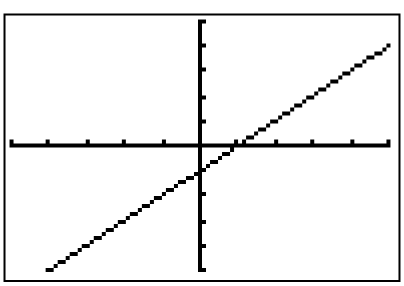
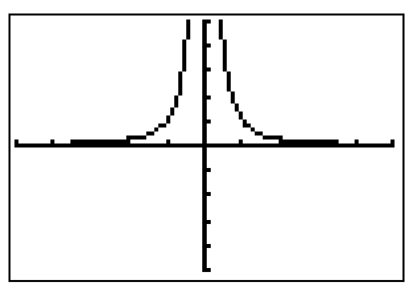
5.



$x = -1$ $y =$

____

6.



(Challenge)

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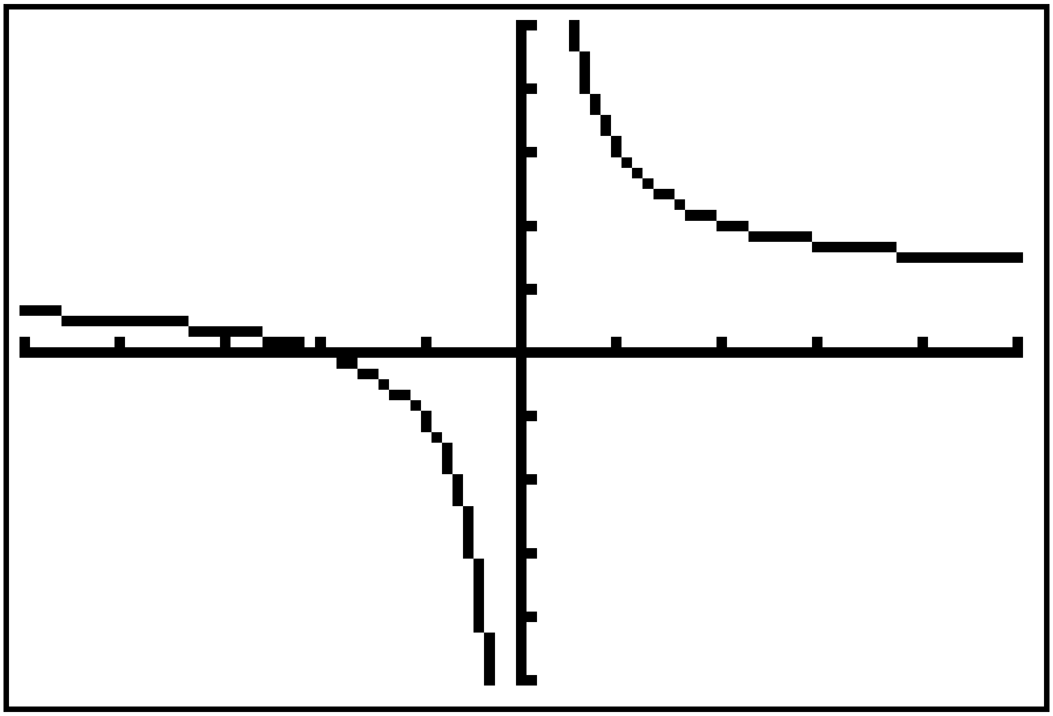
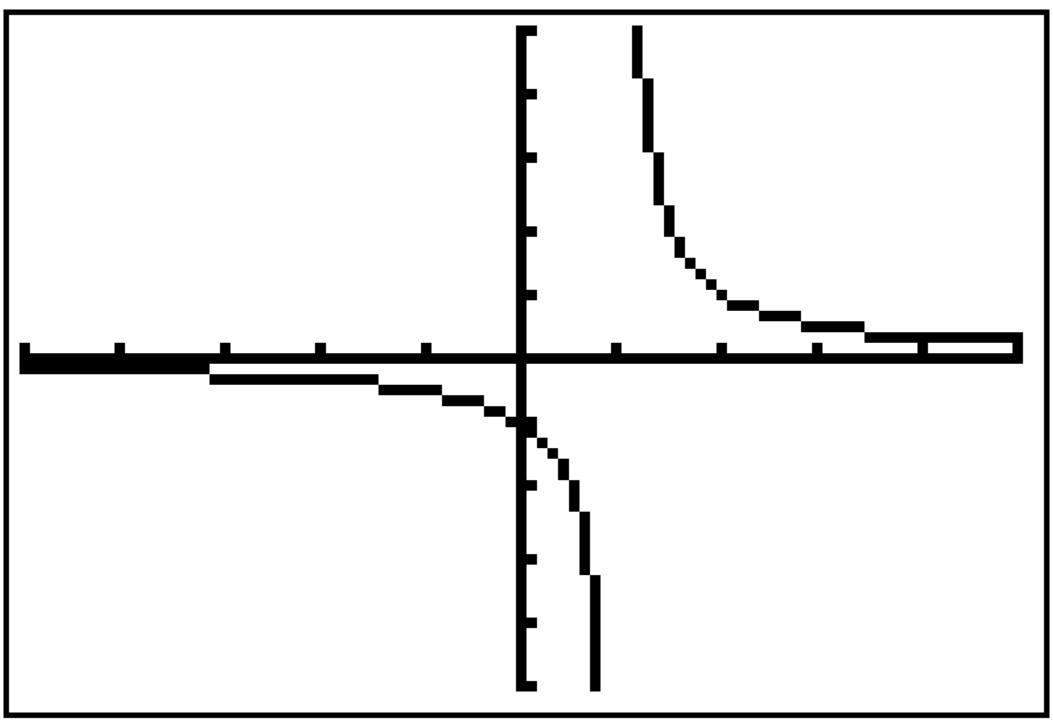
What's My Function? Graphs

1.

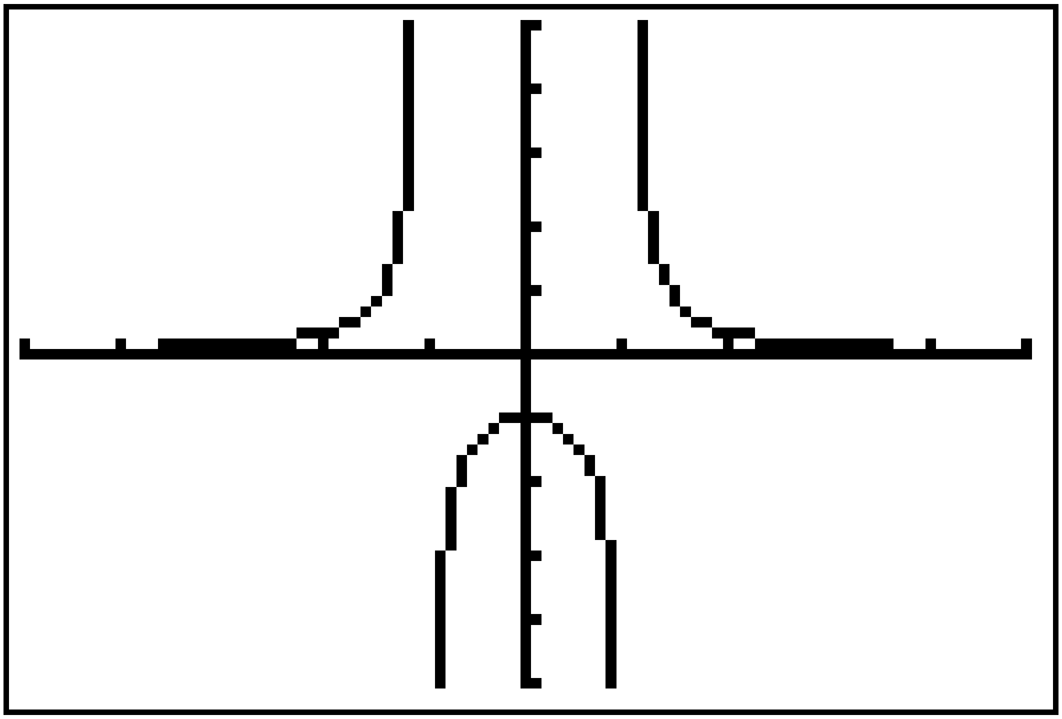


2.



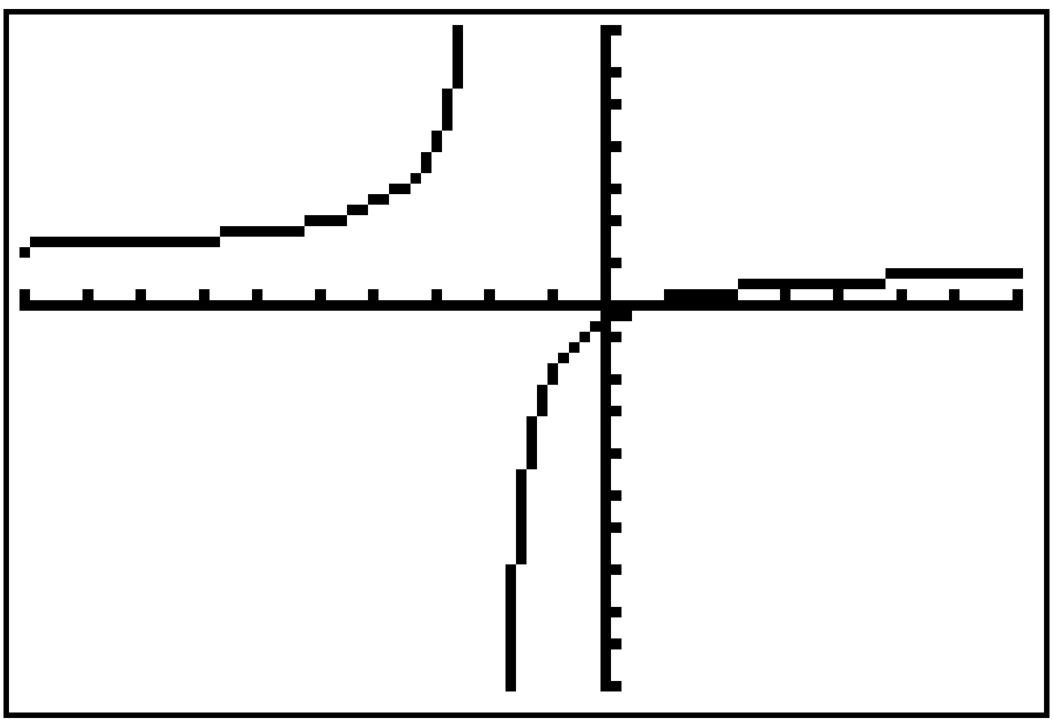
$x = -4$ $y =$

3.



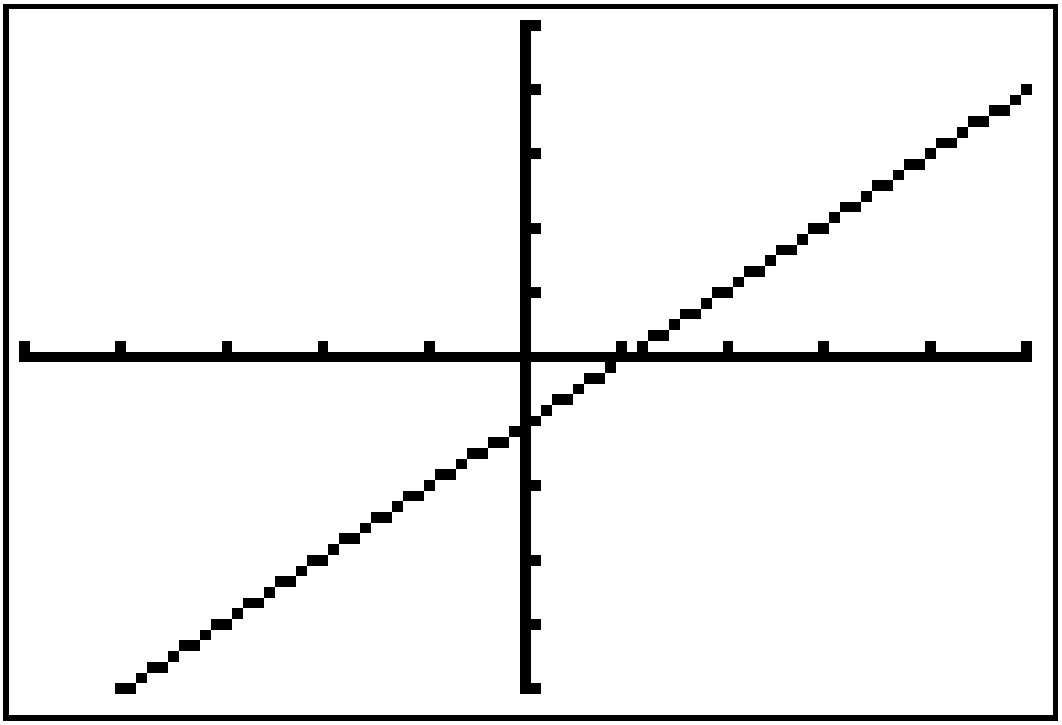
$x = 0$ $y =$

4.



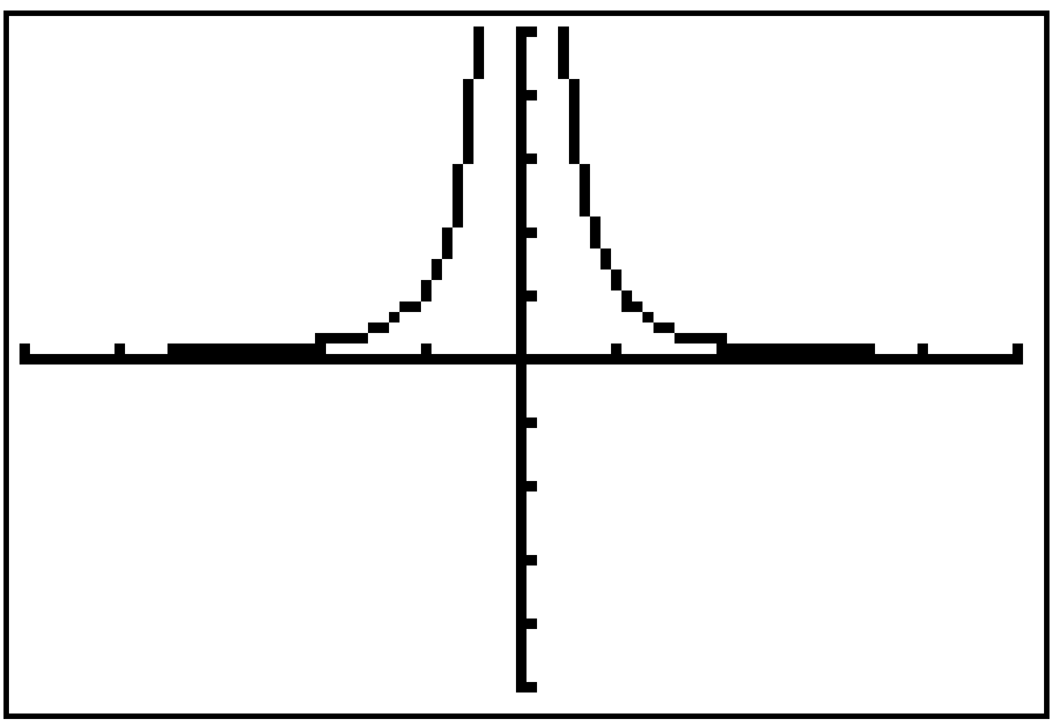
$x = 1$ $y = 0$

5.



$x = -1$ $y =$

6.



(Challenge)

Rational Functions

$$y = \frac{x+3}{x+1}$$

Rational Functions

$$y = \frac{x+3}{x^2}$$

Rational Functions

$$y = \frac{3x^2 - 5x - 12}{x - 3}$$

Rational Functions

$$y = \frac{x^2 - x - 12}{2x^2 - 32}$$

Rational Functions

$$5xy = 10$$

Rational Functions

$$y = 2^x$$

Answer Cards

Copy cards on cardstock, and cut out.

None	$x = 0$
None	$y = 0$

Answer Cards

None	None
$(0,1)$	$y = 0$

Answer Cards

None	$x = -4$
None	$(-\frac{4}{3}, 0)$

Answer Cards

$(-3,0)$	$(0,3)$
$x=-1$	$y=1$

Answer Cards

None	$y = 0$
$x = 0$	$(-3,0)$

Answer Cards

None	(0,4)
None	None

Answer Cards

Where $x = 3$	Where $x = 4$
None	$y = \frac{1}{2}$

Answer Cards

$$(-3,0)$$

$$\left(0,\frac{3}{8}\right)$$