Inverse Functions

Reporting Category Functions

Topic Exploring inverse functions

Primary SOL AII.7g The student will investigate and analyze functions algebraically and graphically. Key concepts include inverse of a function. Graphing calculators will be used as a tool to assist in investigation of functions.

Related SOL AII.7a,g

Materials
- Graphing calculators
- Graph paper

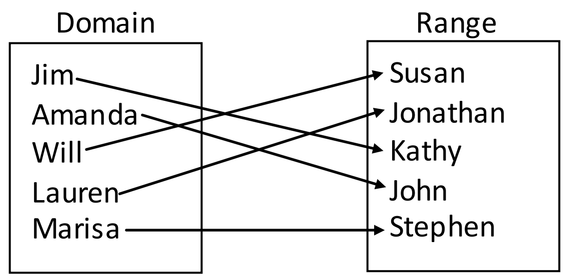
Vocabulary

domain, range, input, output (earlier grades)

inverse function, composition of functions, one-to-one function (AII.7g, h)

Student/Teacher Actions (what students and teachers should be doing to facilitate learning)

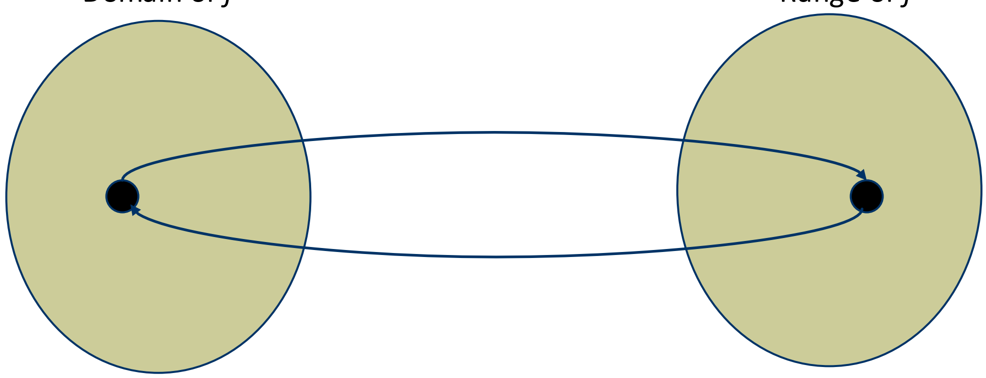
1. Define the inverse of a function as a set of ordered pairs in which
   if $(a,b) \in f$, then $(b,a) \in$ inverse of f.
2. Using the definition above, have students find the inverse of the function below, where the elements of the domain are friends of Monica, and the range consists of the friends' spouses. Ask whether the inverse is a function.



3. Display the function {(−3, 27) (−2, −8) (−1, −1) (0, 0) (1, 1) (2, 8) (3, 27)}, and have students find the inverse. Ask whether the inverse is a function. Then, have them find the inverse of the function {(−3, 9) (−2, 4) (−1, 1) (0, 0) (1, 1) (2, 4) (3, 9)}, and again ask whether the inverse is a function.
4. Ask students how they can tell by looking at a function if its inverse will also be a function. Lead them to see that functions in which the unique inputs (values of x from the domain) correspond or "map" to unique outputs (values of $f(x)$ from the range) have an inverse that is a function. Inform them that this is called a *one-to-one function.*
5. Discuss the vertical line test—what it is and why it works. Ask whether a horizontal line test would tell us anything useful. Have students graph $y = -0.5(x + 2)^2 + 3$ and then graph the

horizontal line $y = 2$. Ask, "Where does the line intersect the parabola? Why does the horizontal line test work?"

6. Explain that a function $y = f(x)$ is a rule that tells us to do something with the input, x. Hence, $f(x) = 2x$ tells us to multiply the input by 2. Emphasize that the inverse of f "undoes" whatever f does; therefore, $f^{-1}(x) = \frac{1}{2}x$, because $\frac{1}{2}$ is the multiplicative inverse of 2. Show students that to find the inverse of $f(x) = 3x - 1$, you perform the inverse operations in reverse order. That is, begin with your input, x, add 1, then divide by 3: $f^{-1}\frac{x+1}{3}$.
7. Demonstrate that for a function $y = f(x)$ to have an inverse function, f must be *one-to-one*: that is, for every x in the domain, there is exactly one y in its range, and likewise, each y in the range corresponds to exactly one x in the domain. The correspondence from the range of f onto the domain of f is, therefore, also a function. It is this function that is the inverse of f.



$$\boxed{\text{Input } x} \xrightarrow{\text{Apply } f(x)} \boxed{f(x)} \xrightarrow{\text{Apply } f^{-1}(x)} \boxed{x}$$

$$f^{-1}[f(x)] = x$$

$$\boxed{\text{Input } x} \xrightarrow{\text{Apply } f^{-1}(x)} \boxed{f^{-1}(x)} \xrightarrow{\text{Apply } f(x)} \boxed{x}$$

$$f[f^{-1}(x)] = x$$

8. Ask students to determine whether each pair of functions below is an inverse of one another.

$h(x) = x^3$	and	$g(x) = \sqrt[3]{x}$
$h[g(x)] = x$	and	$g[h(x)] = x$
$h(\sqrt[3]{x}) = x$	and	$g(x^3) = x$
$(\sqrt[3]{x})^3 = x$	and	$\sqrt[3]{x^3} = x$

   Students should find that they are indeed inverses. Tell them that because the pairs are inverses of one another, we say that $g(x) = h^{-1}(x)$ and $h(x) = g^{-1}(x)$.

9. Have students enter $Y_1 = x^3$ and $Y_2 = \sqrt[3]{x}$ in their graphing calculators and set WINDOW so that $-3 \le x \le 3$ and $-2 \le y \le 2$. Ask what they observe about the graph. Then, have them enter $Y_3 = x$. Have them note the symmetry of the inverse functions with respect to $y = x$. Tell them to think about reflections. Using the TABLE function, point out that if (a,b) is on the graph of *f,* then (b,a) is on the graph of f^{-1}.
10. Direct students to graph the points (−2,−1) (−1,0) (2,1) on a set of coordinate axes, connect the points with line segments, graph the inverse, and then graph $y = x$.
11. Finding the inverse: Inform students that the graph of a one-to-one function, *f*, and its inverse are symmetric with respect to $y = x$. Therefore, we can identify f^{-1} by interchanging the roles of *x* and *y*: if *f* is defined by the equation $y = f(x)$, then f^{-1} is defined by the equation $x = f(y)$. The equation $x = f(y)$ defines f^{-1} implicitly. Solving for *y* will produce the explicit form of f^{-1} as $y = f^{-1}(x)$.
12. Give students the function $f(x) = 2x + 3$, and ask whether *f* is one-to-one. (Yes, it is linear and increasing.)

    $f(x) = 2x + 3$
    $y = 2x + 3$
    $x = 2y + 3$

    The variables *x* and *y* have been interchanged in the original equation. This equation defines f^{-1} implicitly. Solving for *y*,

    $2y + 3 = x$
    $2y = x - 3$
    $y = \frac{1}{2}(x - 3)$
    $f^{-1}(x) = \frac{1}{2}(x - 3)$ This is the explicit form.

    What is the domain of *f* ? What is the domain of f^{-1}?
    What is the range of *f* ? What is the range of f^{-1}?
13. Have students graph $Y_1 = f(x) = 2x + 3$ and its inverse, $Y_2 = f^{-1}(x) = \frac{1}{2}(x - 3)$, with $Y_3 = x$. Point out the symmetry of the graphs with respect to $y = x$.

Assessment

- **Questions**
  - On graph paper, graph the functions $f(x) = \frac{3}{4}x + 2$ and $g(x) = \frac{4}{3}x - 2$. Are these functions inverses of one another? Why, or why not?
  - Graph $f(x) = -3x + 5$. Now graph $y = x$. Without manipulating the original function algebraically, graph $f^{-1}(x)$. Use the line of reflection to determine points on the inverse function.
- **Journal/Writing Prompts**
  - Describe, in detail, three methods for finding the inverse of $f(x) = \frac{2}{3}x - 4$.

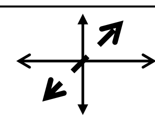
- Explain what it means for a function to be "one-to-one," and describe two methods for determining whether or not a function is one-to-one.
- Justify the identity function, $y = x$, being the line of reflection for a function and its inverse.

Extensions and Connections (for all students)

- Have students find the inverse of $y = f(x) = x^2$. Because $f(x)$ is not one-to-one on its domain, restrict the domain to $x \geq 0$.
- Have students graph $y = 10^x$ and $y = \log x$. Ask, "Are both functions one-to-one? Are they mirror images of one another? If so, with respect to which line? What can we conclude about the two functions?"
- Pose the following problem: "We have studied many functions, including absolute value, quadratic, square root, and cube root. Of those functions, identify which have inverses and which do not. Explain your reasoning."

Strategies for Differentiation

- Have students play a matching game to match inverse functions, both as graphs and as algebraic expressions. Use either cards or an interactive whiteboard.
- Use the following graphic organizer to remind students of the key concepts and processes for inverses.

$\in$	
$1-1$	$f \circ f^{-1}$
SWAP	

Go over the meaning of the above, as follows:

Elements of the inverse of a function are determined by $(a,b) \in f \Rightarrow (b,a) \in \text{INV of } f$	The graph of a function and its inverse are symmetric to the line $y = x$.
The inverse of f is a function only if f is a one-to-one function.	To prove two functions are inverses of one another, show their composition is the identity function: $f \circ f^{-1}(x) = x$
To find the inverse of a function, **SWAP** x and y, and solve for y.	