Composition of Functions

Reporting Category    Functions
Topic                Exploring composition of functions
Primary SOL          All.7h  The student will investigate and analyze functions algebraically and graphically. Key concepts include composition of multiple functions. Graphing calculators will be used as a tool to assist in investigation of functions.

Related SOL          All.7a, g

Materials
- Graphing calculators
- Attached handout

Vocabulary
- domain, range, input, output (earlier grades)
- composite functions, composition of functions, inverse functions (All.7g, h)

Student/Teacher Actions (what students and teachers should be doing to facilitate learning)

1. Present the following explanation to students:
   Consider the function \( h(x) = \sqrt{4x} \). Using an input of 9 to evaluate \( h \), we see that \( h(9) = \sqrt{4(9)} = \sqrt{36} = 6 \). So, \( h(9) = 6 \). Since we performed two operations—multiplication and finding the square root—we can think of \( h \) as a composite of two functions. Let’s call these two functions \( f(x) \) and \( g(x) \), with \( f(x) = 4x \) and \( g(x) = \sqrt{x} \). We can evaluate \( h(9) \) by finding the output of \( f(9) \) and using that output as the input of function \( g \).
   - First, use 9 as the input for \( f \), and find \( f(9) = 4(9) = 36 \).
   - Next, use that output as the input for \( g \) and find \( g(36) = \sqrt{36} = 6 \).
   - Therefore, \( h(9) = g \circ f(9) = g(f(9)) = g(36) = 6 \).

When we use an output of one function as an input for another function, we are creating a composition of functions. In our example, \( h \) is a composition of \( f \) and \( g \), which is written as \( h(x) = g(f(x)) \) or \( h(x) = g \circ f(x) \). Both equations are read “\( h(x) \) equals \( g \) of \( f \) of \( x \).”

2. Demonstrate how an input-output diagram can be used to model a composition of functions. Stress the fact that inputs are domain values, whereas outputs represent range values. Display an input-output diagram similar to the one at right to illustrate \( g(f(3)) \) for \( f(x) = x + 1 \) and \( g(x) = -2x \).
3. After guiding students through several examples, distribute copies of the attached Compositions of Functions handout, and have them complete it.
4. When students are comfortable with evaluating compositions of functions, hold a discussion about numbers 9 and 10 on the handout. Be sure students notice that \( f \circ g(x) \) and \( g \circ f(x) \) are the same. Ask why this is. If students do not notice it without guidance, suggest that they look closely at the original functions and the operations involved. Hopefully, they will realize that the operations used are inverse operations and are performed in reverse order in one function compared to the other.
5. Introduce inverse functions: when \( f[g(x)] = g[f(x)] \), \( f \) and \( g \) are said to be inverse functions.

Assessment

- Questions

\[
\begin{array}{c|c}
\text{g} & \text{h} \\
\hline
1 & 0 \\
2 & 3 \\
3 & 0 \\
4 & 9 \\
\end{array}
\]

- In the mapping above, what are all points on the graph of \( h[g(x)] \) ?
- In the mapping above, what are the domain and range of \( g[h(x)] \) ?
- Given \( g(x) = x^2 \) and \( h(x) = x - 3 \), draw an input-output diagram to illustrate \( g \circ h(-4) \). What is the output?
- Given \( f(x) = \sqrt{x} \), \( g(x) = (x + 1)^2 \), and \( h(x) = -2x \), evaluate the following:
  - \( f[g(x)] \)
  - \( h \circ f(8) \)
  - \( g[h(m)] \)
  - \( f \circ h(-2) \)
  - \( g \circ f \circ h(-8) \)

- Journal/Writing Prompts
  - Explain how to determine the output of a composition of functions, given the input.
  - When using a mapping to illustrate a composition of functions, describe a map in which there would be no output for an input value. Also, describe a map that would have the same number of outputs as inputs when the composition is performed.

Extensions and Connections (for all students)

- Have students graph \( f(x) = 4x \) and \( g(x) = \sqrt{x} \) simultaneously on their calculators. Then, have them graph \( f[g(x)] \) on the same screen. Ask what they notice. Next, have them add the graph of \( g[f(x)] \). Discuss why the general shapes of both compositions are the same.

Strategies for Differentiation

- Have students color-code input and output values.
• Create a handout of problems for students to work individually and then exchange with partners to check each other’s work.
• Create “function machines” for students to use to produce outputs for given inputs. Then, have them exchange machines and determine the functions that were used to generate the outputs.
• Conduct a thorough review of evaluating functions for given input values.
Compositions of Functions

Given \( f(x) = -2x + 1 \), \( g(x) = x^2 \), and \( h(x) = -\frac{1}{2}x + \frac{1}{2} \), evaluate the following:

1. \( f(-6) \)  
2. \( g(-3) \)  
3. \( h(4) \)

4. \( f(g(2)) \)  
5. \( h[g(8)] \)  
6. \( g \circ f(5) \)

7. \( g \circ h(7) \)  
8. \( f[g(c)] \)  
9. \( f[h(5)] \)

10. \( h[f(r)] \)  
11. \( h[g(f(3))] \)  
12. \( f \circ g \circ h(3) \)

Refer to the maps below to determine the domain and range for each of \( g \circ f(x) \) and \( f \circ g(x) \). Name the points included in each composition of functions.

13. \( g \circ f(x) \)

\[
\begin{array}{c}
\text{domain} \quad \text{range} \quad \text{points} \\
\text{domain} \quad \text{range} \quad \text{points}
\end{array}
\]

\[f\]

\[
\begin{array}{c|c}
0 & -4 \\
2 & 0 \\
3 & 2 \\
6 & 8 \\
\end{array}
\]

\[g\]

\[
\begin{array}{c|c}
0 & 0 \\
1 & 1 \\
2 & 4 \\
3 & 9 \\
\end{array}
\]

14. \( f \circ g(x) \)

\[
\begin{array}{c}
\text{domain} \quad \text{range} \quad \text{points} \\
\text{domain} \quad \text{range} \quad \text{points}
\end{array}
\]

\[f\]

\[
\begin{array}{c|c}
3 & -1 \\
4 & -7 \\
5 & 0 \\
6 & 4 \\
\end{array}
\]

\[g\]

\[
\begin{array}{c|c}
0 & 1 \\
2 & 2 \\
4 & 7 \\
6 & 8 \\
\end{array}
\]
15. \( g \circ f(x) \)  \( f \circ g(x) \)

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**Domain**

- \( g \circ f(x) \):
  - \( x \) values: 0, 1, 3, 4
  - Domain: _____________

- \( f \circ g(x) \):
  - \( x \) values: -2, -1, 1, 2
  - Domain: _____________

**Range**

- \( g \circ f(x) \):
  - \( y \) values: -2, -1, 1, 2
  - Range: _____________

- \( f \circ g(x) \):
  - \( y \) values: 0, 1, 3, 4
  - Range: _____________

**Points**

- \( g \circ f(x) \):
  - Points: ______________

- \( f \circ g(x) \):
  - Points: ______________