Instructional Guide to Support 2023 Geometry *Standards of Learning*

Virginia Department of Education

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Introduction

The Mathematics Instructional Guide, a companion document to the 2023 Mathematics *Standards of Learning*, amplifies the Standards of Learning by defining the core knowledge and skills in practice, supporting teachers and their instruction, and serving to transition classroom instruction from the 2016 *Mathematics Standards of Learning* to the newly adopted 2023 Mathematics *Standards of Learning*. Instructional supports are accessible in #GoOpenVA and support the decisions local school divisions must make concerning local curriculum development and how best to help students meet the goals of the standards. The local curriculum should include a variety of information sources, readings, learning experiences, and forms of assessment selected at the local level to create a rigorous instructional program.

For a complete list of the changes by standard, the 2023 Virginia Mathematics *Standards of Learning – Overview of Revisions* is available and delineates in greater specificity the changes for each grade level and course.

The Instructional Guide is divided into three sections: Understanding the Standard, Skills in Practice, and Concepts and Connections aligned to the Standard. The purpose of each is explained below.

Understanding the Standard

This section includes mathematics understandings and key concepts that assist teachers in planning standards-focused instruction. The statements may provide definitions, explanations, or examples regarding information sources that support the content. They describe what students should know (core knowledge) as a result of the instruction specific to the course/grade level and include evidence-based practices to approaching the Standard. There are also possible misconceptions and common student errors for each standard to help teachers plan their instruction.

Skills in Practice

This section outlines supporting questions and skills that are specifically linked to the standard. They frame student inquiry, promote critical thinking, and assist in learning transfer. Curriculum writers and teachers should use them to plan instruction to deepen understanding of the broader unit and course objectives. This is not meant to be an exhaustive list of student expectations.

Concepts and Connections

This section outlines concepts that transcend grade levels and thread through the K through 12 mathematics program as appropriate at each level. Concept connections reflect connections to prior grade-level concepts as content and practices build within the discipline as well as potential connections across disciplines.

Geometry

Reasoning, Lines, and Transformations

Reasoning is the foundation for problem solving, higher-order thinking, understanding, and application. Students use reasoning skills in every branch of mathematics in addition to other courses. These skills unpack building blocks that are necessary for advanced mathematical thinking, understanding, and application. Also, mastery of contextual application of lines and transformations prepare students for engineering, calculus, and physical sciences.

Throughout Geometry, students will translate, construct, and judge the validity of a logical argument and use and interpret Venn diagrams. Additionally, students will analyze the relationships of parallel lines cut by a transversal, and solve problems including contextual problems, involving symmetry and transformation.

G.RLT.1

The student will translate logic statements, identify conditional statements, and use and interpret Venn diagrams.

Students will demonstrate the following Knowledge and Skills:

a) Translate propositional statements and compound statements into symbolic form, including negations ($\sim p$, read "*not p*"), conjunctions ($p \wedge q$, read "*p and q*"), disjunctions ($p \vee q$, read "*p or q*"), conditionals ($p \rightarrow q$, read "*if p then q*"), and biconditionals ($p \leftrightarrow q$, read "*p if and only if q*"), including statements representing geometric relationships.

b) Identify and determine the validity of the converse, inverse, and contrapositive of a conditional statement, and recognize the connection between a biconditional statement and a true conditional statement with a true converse, including statements representing geometric relationships.

c) Use Venn diagrams to represent set relationships, including union, intersection, subset, and negation.

d) Interpret Venn diagrams, including those representing contextual situations.

Understanding the Standard

- Symbolic notation is used to represent logical arguments, including the use of $\rightarrow$, $\leftrightarrow$, $\sim$, $\therefore$, $\wedge$, and $\vee$.

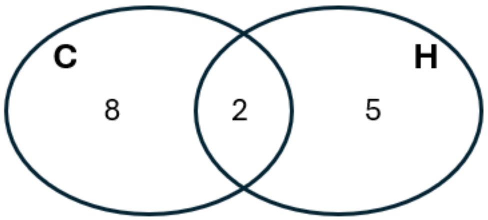
$\vee$	or
$\wedge$	and
$\rightarrow$	read "implies", if… then…
$\leftrightarrow$	read "if and only if"
iff	read "if and only if"
~	not
$\therefore$	therefore

- Logical arguments consist of a set of premises or hypotheses and a conclusion.
- A conditional statement is a logical argument consisting of a set of premises, hypothesis (p), and conclusion (q).
  - If p, then q or *p*→*q*
  - If an angle is a right angle, then its measure is 90°.
- A converse is formed by interchanging the hypothesis and conclusion of a conditional statement.
  - *If q, then p or q*→*p*)
  - If an angle measures 90°, then the angle is a right angle.
- An inverse is formed by negating the hypothesis and conclusion of a conditional statement.
  - If $\sim p$, then $\sim q$ or $\sim p \rightarrow \sim q$
  - If an angle is not a right angle, then its measure is not 90°.
- A contrapositive is formed by interchanging and negating the hypothesis and conclusion of a conditional statement.
  - If $\sim q$, then $\sim p$ or $\sim q \rightarrow \sim p$
  - If an angle does not measure 90°, then the angle is not a right angle.
- When a conditional (*p*→*q*) and its converse (*q*→*p*) are true, the statements can be written as a biconditional:
  - *p iff q*
  - *p if and only if q*
  - *p*↔*q*
  - The Pythagorean Theorem and its converse can be used as an example: *If a triangle is a right triangle, then the sum of the squares of the legs is equal to the square of the hypotenuse* ($a^2 + b^2 = c^2$). *If the sum of the squares of the legs is equal to the square of the hypotenuse* ($a^2 + b^2 = c^2$), *then the triangle is a right triangle. Therefore, a triangle is a right triangle if and only if the sum of the squares of the legs is equal to the square of the hypotenuse* ($a^2 + b^2 = c^2$).

- Truth and validity are not synonymous. Valid logical arguments may be false. Validity requires only logical consistency between the statements, but it does not imply true statements.
- For example, the following argument is valid, but not true: *If you are a happy person, then you like animals. If you like animals, then you like dogs. Therefore, if you are a happy person, then you like dogs.*
- Formal proofs utilize symbols of formal logic to determine the validity of a logical argument.
- Inductive reasoning, deductive reasoning, and proofs are critical in establishing general claims.
- Inductive reasoning is the method of drawing conclusions from a limited set of observations.
- Deductive reasoning is the method that uses logic to draw conclusions based on definitions, postulates, and theorems. Valid forms of deductive reasoning include the law of syllogism, the law of contrapositive, the law of detachment, and the identification of a counterexample.
- Proof is a justification that is logically valid and based on initial assumptions, definitions, postulates, theorems, and/or properties.
- The law of detachment states that if $p \rightarrow q$ is true and p is true, then q is true. For example, if two angles are vertical, then they are congruent. ∠A and ∠B are vertical, therefore ∠A ≅ ∠B.
- The law of syllogism states that if $p \rightarrow q$ is true and $q \rightarrow r$ is true, then $p \rightarrow r$ is true. For example, if two angles are vertical, then they are congruent. If two angles are congruent, then they have the same measure. Thus, if two angles are vertical, then they have the same measure.
- The law of contrapositive states that if a conditional statement ($p \rightarrow q$) is true, then its contrapositive ($\sim q \rightarrow \sim p$) is also true. For example, if two angles are vertical, then they are congruent. ∠A ≇ ∠B, therefore ∠A and ∠B are not vertical.
- A counterexample is used to show an argument is false. For example, the argument "*All rectangles are squares,*" is proven false with the following counterexample since quadrilateral *ABCD* is a rectangle but not a square.

- A counterexample of a statement confirms the hypothesis but negates the conclusion.
- Exploration of the representation of conditional statements using Venn diagrams may assist in deepening student understanding.

- Venn diagrams can be interpreted within contextual situations.
- Venn diagrams can be used to support the understanding of special quadrilateral relationships or problems involving probability.
  - Surveys can provide opportunities for discussion of experimental probability. The Venn diagram below shows the results of a survey of students to determine who likes comedy movies (C) and/or horror movies (H). Eight students like comedy movies, but not horror movies; five students like horror movies, but not comedy movies; and two students like both comedy movies and horror movies.



Skills in Practice

While the five process goals are expected to be embedded in each standard, the Skills in Practice highlight the most prevalent process goals in relation to the content presented.

Mathematical Representations: Students will benefit from additional practice using and interpreting logic symbols $\therefore$, $\wedge$ and $\vee$.

Let *m* represent: *Angle A is obtuse*.
Let *n* represent: *Angle B is obtuse*.

Which is a symbolic representation of the following argument?

Angle A is obtuse if and only if Angle B is obtuse.

Angle A is obtuse or Angle B is obtuse.

Therefore, Angle A is obtuse and Angle B is obtuse.

A.	B.	C.	D.
$m \rightarrow n$ $m \wedge n$ $\therefore m \vee n$	$m \rightarrow n$ $m \vee n$ $\therefore m \wedge n$	$m \leftrightarrow n$ $m \wedge n$ $\therefore m \vee n$	$m \leftrightarrow n$ $m \vee n$ $\therefore m \wedge n$

Misconceptions occur when students do not fully understand the meaning of each logic symbol. They tend to make mistakes when converting between symbolic form and written statements. Common errors include using $\leftrightarrow$ in place of $\rightarrow$ or confusing $\wedge$ and $\vee$. *The correct answer to this problem is* D.

Mathematical Reasoning: Students may need practice judging the validity of a logical argument and using valid forms of deductive reasoning, including the law of syllogism, the law of contrapositive, the law of detachment, and counterexamples.

Let *p* = "a dog eats bread"
Let *q* = "the dog gains weight"

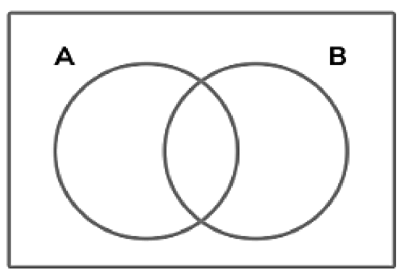
$p \rightarrow q$, "If a dog eats bread, then the dog gains weight" is a true statement.

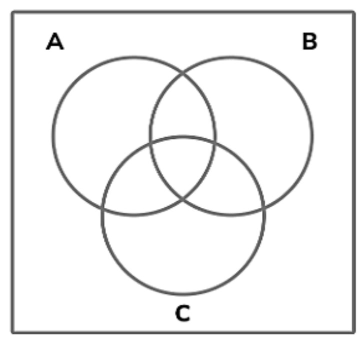
John's dog eats bread. What can be concluded?

Students may incorrectly apply geometric rules or properties to situations where they don't apply, leading to errors in problem-solving. *Since John has a dog who eats bread and the conditional stated "If a dog eats bread, then the dog gains weight," it can be concluded John's dog gains weight. Using the Law of Detachment, p, John's dog eats bread, is true, thus q, John's dog gains weight is true.*

Mathematical Connections: Students may have difficulty using Venn diagrams to represent set relationships and contextual situations.

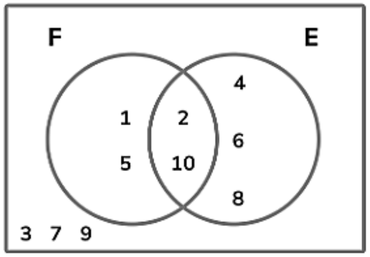
- Venn diagrams provide a visual representation of two or more sets. The following are examples of a two set and a three set Venn diagram with the following features –





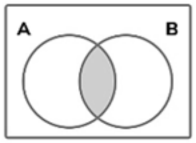
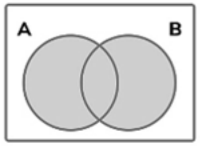
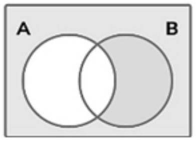
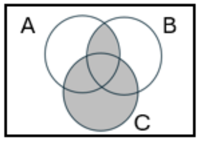
- The universal set is a rectangle outlining the space in which all values within the smaller sets are found.
- The set A, shown using a circle and labeled A.
- The set B, shown using a circle and labeled B.
- The set C, shown using a circle and labeled C (reference the three set Venn diagram).
- Set A and set B (and set C) overlap, showing the shared parts of set A, set B, (and set C). This is called the intersection.
- To analyze data using Venn diagrams, all of the values within each set must be correctly allocated into the correct part of the Venn diagram. The number of sets is usually outlined or logically deduced from the information provided.

- When constructing a Venn diagram, draw a region containing two or more overlapping circles (or ellipses), each representing a set. Then, either fill in the information that is given or what can be logically deduced. For example, the Venn diagram below shows the set of numbers {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} that have been sorted into factors of 10 (set F) and even numbers (set E).

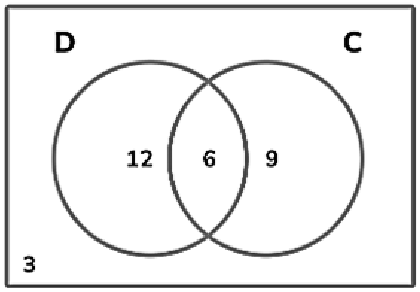


All numbers 1 through 10 are represented on the Venn diagram. The factors of 10 appear within set F; and even numbers appear within set E. The numbers that are both factors of 10 and even numbers appear in the intersection of sets F and E. The numbers that are not factors of 10 or even numbers are outside of set F and set E but remain part of the universal set of numbers 1-10.

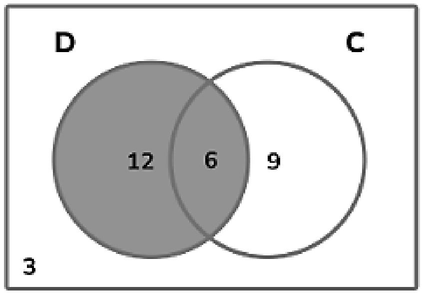
- To describe a subset, students need to understand key symbols and set notation for different sets including the intersection of sets, the union of sets and the absolute complement of sets. The following are examples of set notation with appropriate symbols and their meanings –

$A \cap B$	'A and B' The intersection of A and B. The elements in both sets A and B.	
$A \cup B$	'A or B' The union of A and B. Any element in set A or set B or both.	
$\sim A$	'Not A ' The complement of A. Any element not in A.	
$C \cup (A \cap B)$	The union of C and the intersection of A and B. The elements in both A and B or C.	

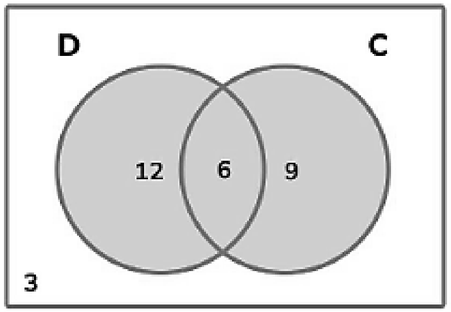
- To calculate the number of items in a subset of a Venn diagram, add together the frequencies of the required subset. The following Venn diagram shows the number of people who own a cat (C) or a dog (D).



- How many people own a dog? Some students will say that only 12 people own a dog. These students have neglected to include 6, the intersection of people owning both a dog and a cat. Another common misconception is for students to add 12 + 6 + 3, including those who own neither a dog nor cat. Teachers may suggest the strategy of shading set D (dog owners) to help students see what numbers are included. See the illustration below.



- How many people own a dog or a cat? A common misconception is for students to add all the numbers 12 + 6 + 9 + 3, thinking the entire diagram represents only dog and cat owners. This shows a misunderstanding of the universal set, and that 3 people are not dog or cat owners.



Concepts and Connections

CONCEPTS

Logic and reasoning provide the foundation of how we use explanations and justifications. Proofs are developed through using higher-order thinking and logical statements.

CONNECTIONS

- *Within the grade level/course*:
  - Throughout each Geometry SOL, students will apply logic and reasoning skills to prove, justify, or confirm answers using postulates, theorems, definitions, and other appropriate justification statements.
- *Vertical Progression*:
  - Critical thinking skills have been embedded in each Mathematics SOL prior to Geometry. Beyond Geometry, students will continue to apply logic and reasoning skills.
  - A.EO.1 The student will represent verbal quantitative situations algebraically and evaluate these expressions for given replacement values of the variables.

Resources to Support Local Curriculum

- A list of approved textbooks and instructional materials is posted on the VDOE website.
- EOC Geometry Formula Sheet (PDF)

G.RLT.2

The student will analyze, prove, and justify the relationships of parallel lines cut by a transversal.

Students will demonstrate the following Knowledge and Skills:

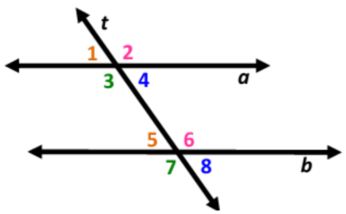
a) Prove and justify angle pair relationships formed by two parallel lines and a transversal, including:
   i) corresponding angles;
   ii) alternate interior angles;
   iii) alternate exterior angles;
   iv) same-side (consecutive) interior angles; and
   v) same-side (consecutive) exterior angles.

b) Prove two or more lines are parallel given angle measurements expressed numerically or algebraically.

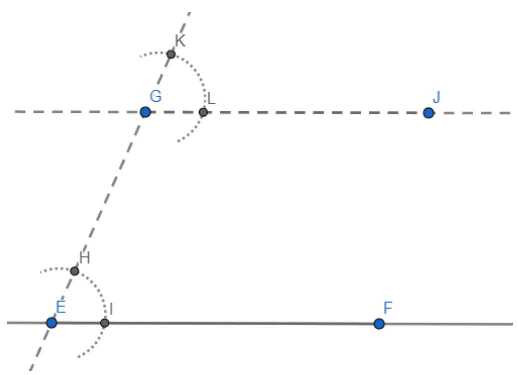
c) Solve problems by using the relationships between pairs of angles formed by the intersection of two parallel lines and a transversal.

Understanding the Standard

- Parallel lines are coplanar lines that do not intersect. Parallel lines have the same slope.
- A transversal is a line that intersects at least two other lines.
- Parallel lines intersected by a transversal form angles with specific relationships.
    - Corresponding angles are in matching positions when a transversal crosses at least two lines. In the model below ∠2 and ∠6 are corresponding angles.
    - Alternate interior angles are inside the parallel lines and on opposite sides of the transversal. In the model below ∠3 and ∠6 are alternate interior angles.
    - Alternate exterior angles are outside the two parallel lines and on opposite sides of the transversal. In the model below ∠2 and ∠7 are alternate exterior angles.
    - Consecutive interior angles are inside the two parallel lines and on the same side of the transversal. In the model below ∠4 and ∠6 are consecutive interior angles.
    - Consecutive exterior angles are outside the parallel lines and on the same side of the transversal. In the model below ∠1 and ∠7 are consecutive exterior angles.



- If two parallel lines are intersected by a transversal, then:
    - corresponding angles are congruent
    - alternate interior angles are congruent;
    - alternate exterior angles are congruent;
    - consecutive (same-side) interior angles are supplementary; and
    - consecutive (same-side) exterior angles are supplementary.
- Transformations of vertical and linear angle pairs can be used to explore relationships of alternate interior, alternate exterior, corresponding, and same-side interior angles.
- To prove two or more lines parallel, one of the angle pairs listed above must be shown to be true. The angles must be on the same transversal that intersects both or all of the lines.
- The parallel line construction uses the Converse of the Corresponding Angles Theorem which states, "If two lines ($\overleftrightarrow{EF}$ and $\overleftrightarrow{GJ}$) and a transversal ($\overleftrightarrow{EG}$) form corresponding angles ($\angle HEI$ and $\angle KGL$) that are congruent, then the lines ($\overleftrightarrow{EF}$ and $\overleftrightarrow{GJ}$) are parallel."

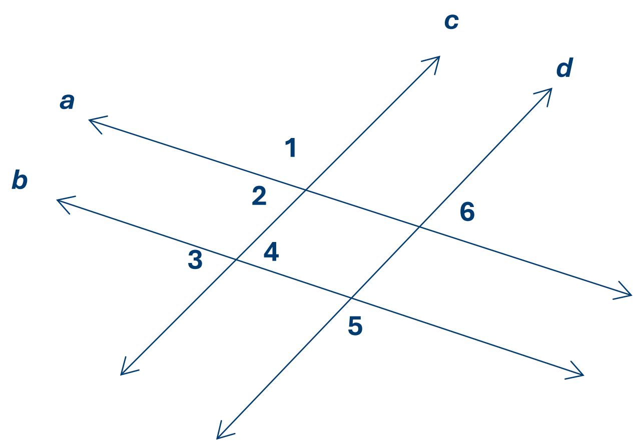


Skills in Practice

While the five process goals are expected to be embedded in each standard, the Skills in Practice highlight the most prevalent process goals in relation to the content presented.

Mathematical Problem Solving: Students may benefit from additional practice identifying which parts of a figure can be used to determine whether lines are parallel when more than one statement may be true.

- Given: Lines *a* and *b* intersect lines *c* and *d*.

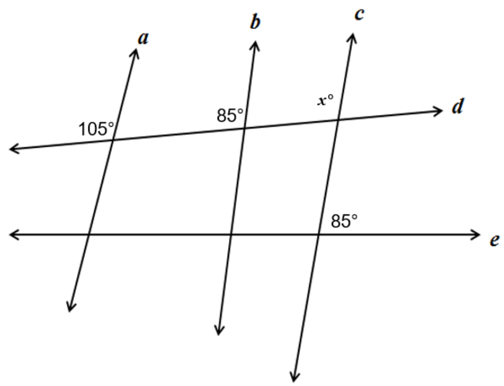


- Which of these statements could be used to prove $a \parallel b$ and $c \parallel d$?
    - A. ∠1 and ∠2 are supplementary and ∠5 ≅ ∠6
    - B. ∠1 ≅ ∠3 and ∠3 ≅ ∠5
    - C. ∠3 and ∠5 are supplementary, and ∠5 and ∠6 are supplementary
    - D. ∠3 ≅ ∠4 and ∠2 ≅ ∠6

A common mistake is when students do not select a statement that applies to appropriate angle pairs from both sets of lines that need to be proven parallel. They must select appropriate pairs of angles from both lines a and b and c and d with appropriate justifications. Some students may select option B. This indicates they understand that ∠1 and ∠3 and ∠5 do have a relationship, but have confused consecutive exterior angles with corresponding angles or have mistakenly thought they were congruent and not supplementary. The correct answer to this problem is: C, ∠3 and ∠5 are supplementary and they are consecutive exterior angles which proves $c \parallel d$; ∠5 and ∠6 are supplementary and they are consecutive exterior angles which proves $a \parallel b$.

Mathematical Reasoning: Students may need more practice determining parallelism when given complex figures. The example shows a complex figure with more than one transversal.

Given: *a* ∥ *c*



- What is the value of *x*?
  Some students may think *x* = 85 degrees, incorrectly assuming *b* ∥ *c*. It is important for students to understand they must use the given information and not make assumptions about a diagram based on how it looks. Teachers may wish to have students use colored pencils to highlight the given parallel lines and transversal that intersects those lines. When parallel lines *a* and *c* are highlighted one colored and transversal line *d* is highlighted a different color, students are able to see the relationship between 105 degrees and *x*.

- Can you prove *b* ∥ *c*? Explain your answer.
  Some students will provide a "yes" or "no" without an explanation. Having students explain or justify their answer will help them analyze the information given and think critically about what is and is not provided in the figure. Lines *b* and *c* are not parallel because the only angles given relative to these lines are the two 85 degree angles. There is not an angle relationship that exists between them.

Students should be reminded to rely on postulates and theorems learned to make judgements regarding parallelism.

Concepts and Connections

Concepts

Lines transcend mathematical content areas. Lines and angles are embedded in every area of mathematics.

CONNECTIONS

- *Within the grade level/course*:
  - G.RLT.1 – The student will translate logic statements, identify conditional statements, and use and interpret Venn diagrams.
  - G.TR.2 – The student will, given information in the form of a figure or statement, prove and justify two triangles are congruent using direct and indirect proofs, and solve problems involving measured attributes of congruent triangles.
  - G.TR.3 – The student will, given information in the form of a figure or statement, prove and justify two triangles are similar using direct and indirect proofs, and solve problems, including those in context, involving measured attributes of similar triangles.
  - G.PC.1 – The student will prove and justify theorems and properties of quadrilaterals and verify and use properties of quadrilaterals to solve problems, including the relationships between the sides, angles, and diagonals.
- *Vertical Progression*:
  - 8.MG.1 – The student will use the relationships among pairs of angles that are vertical angles, adjacent angles, supplementary angles, and complementary angles to determine the measure of unknown angles.
  - A.F.1 – The student will investigate, analyze, and compare linear functions algebraically and graphically, and model linear relationships.

Resources to Support Local Curriculum

- A list of approved textbooks and instructional materials is posted on the VDOE website.
- EOC Geometry Formula Sheet (PDF)
- Parallel Lines and Angle Relationships Exemplar Mathematics Instructional Plan (Word | PDF)

G.RLT.3

The student will solve problems, including contextual problems, involving symmetry and transformation.

Students will demonstrate the following Knowledge and Skills:

a) Locate, count, and draw lines of symmetry given a figure, including figures in context.

b) Determine whether a figure has point symmetry, line symmetry, both, or neither, including figures in context.

c) Given an image or preimage, identify the transformation or combination of transformations that has/have occurred. Transformations include:

   i) translations;

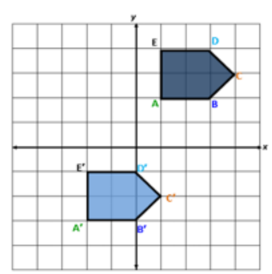
   ii) reflections over any horizontal or vertical line or the lines $y = x$ or $y = -x$;

   iii) clockwise or counterclockwise rotations of 90°, 180°, 270°, or 360° on a coordinate grid where the center of rotation is limited to the origin; and

   iv) dilations, from a fixed point on a coordinate grid.

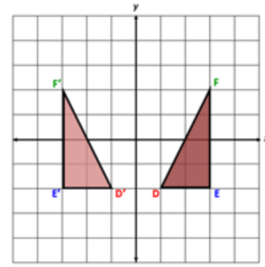
Understanding the Standard

- A transformation of a figure, called a preimage, changes the size, shape, and/or position of the figure to a new figure called the image.
- A rigid transformation (or isometry) is a transformation that does not change the size or shape of a geometric figure. This is a special kind of transformation that does not change the size or shape of a figure.
- The image of an object or function graph after a rigid transformation is congruent to the preimage of the object.
- Congruent figures can be shown through a series of rigid transformations.
- A translation is a rigid transformation in which an image is formed by moving every point on the preimage the same distance in the same direction.



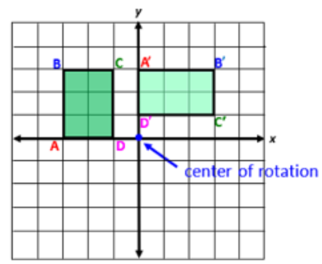
Preimage	Image
A(1,2)	A′(-2,-3)
B(3,2)	B′(0,-3)
C(4,3)	C′(1,-2)
D(3,4)	D′(0,-1)
E(1,4)	E′(-2,-1)

- A reflection is a rigid transformation in which an image is formed by reflecting the preimage over a line called the line of reflection. All corresponding points in the image are equidistant from the line of reflection. The figure in the image below has been reflected over the *y*-axis.



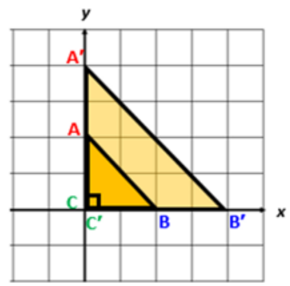
Preimage	Image
D(1,-2)	D′(-1,-2)
E(3,-2)	E′(-3,-2)
F(3,2)	F′(-3,2)

- The midpoint between any set of reflected points lies on the line of reflection.
- The line of reflection can be determined by finding the midpoint (or balance point) between any set of two reflected points.
- A rotation is a rigid transformation in which an image is formed by rotating the preimage about a point called the center of rotation. The center of rotation may or may not be on the preimage. The figure in the image below has been transformed by a 90 degree clockwise rotation about the origin.



Preimage	Image
A(-3,0)	A′(0,3)
B(-3,3)	B′(3,3)
C(-1,3)	C′(3,1)
D(-1,0)	D′(0,1)

- A dilation is a transformation in which an image is formed by enlarging or reducing the preimage proportionally by a scale factor from the center of dilation. The center of dilation may or may not be on the preimage. The image is similar to the preimage. The image below has been dilated by a scale factor of 2.



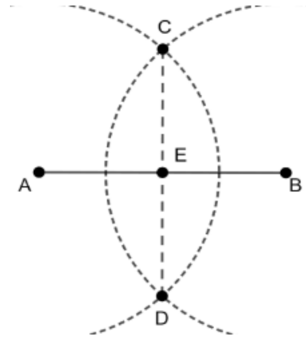
Preimage	Image
A(0,2)	A′(0,4)
B(2,0)	B′(4,0)
C(0,0)	C′(0,0)

- Symmetry and transformations can be explored with coordinate methods.
- Transformations and combinations of transformations can be used to define and describe the movement of objects in a plane or coordinate system.
- The rules for transformations can be described using coordinates and/or verbal descriptions.

Coordinate Transformations	
Translation	$(x, y) \rightarrow (x + a, y + b)$
Reflection across the *x*-axis	$(x, y) \rightarrow (x, -y)$
Reflection across the *y*-axis	$(x, y) \rightarrow (-x, y)$
Reflection across the line $y=x$	$(x, y) \rightarrow (y, x)$
Reflection across the line $y=-x$	$(x, y) \rightarrow (-y, -x)$
Rotation 90° (counterclockwise) about the origin	$(x, y) \rightarrow (-y, x)$
Rotation 180° about the origin	$(x, y) \rightarrow (-x, -y)$
Rotation 270° (counterclockwise) about the origin	$(x, y) \rightarrow (y, -x)$
Dilation with respect to the origin and scale factor of k	$(x, y) \rightarrow (kx, ky)$

- A set of points has line symmetry if and only if there is a line, *l*, such that the reflection through *l* of each point in the set is also a point in the set.
- Point symmetry exists when a figure is built around a single point called the center of the figure. For every point in the figure, there is another point found directly opposite it on the other side of the center, at the same distance from the center. A figure with point symmetry will appear the same after a 180° rotation. In point symmetry, the center point is the midpoint of every segment formed by joining a point to its image.

- The perpendicular bisector construction creates the perpendicular bisector as the line of reflection of the provided line segment.

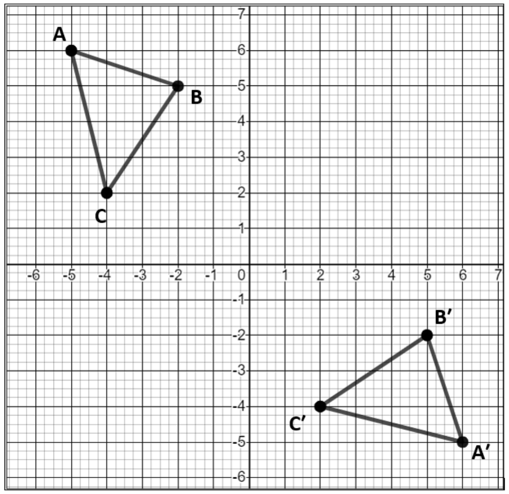


Skills in Practice

While the five process goals are expected to be embedded in each standard, the Skills in Practice highlight the most prevalent process goals in relation to the content presented.

Mathematical Problem Solving: Students may need practice identifying the result of a combination of transformations and completing a combination of transformations to determine the new coordinates of a given figure.

Example: Which sequence of two transformations maps △ABC to △A'B'C'? (The vertices of the triangles have integral coordinates.)



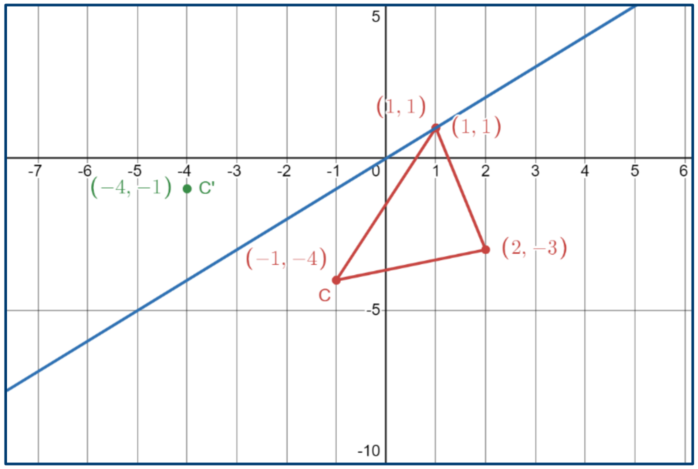
- A rotation 90° counterclockwise about the origin, then a reflection over the line $y = -x$
- A rotation 90° clockwise about the origin, then a reflection over the *x*-axis
- A translation 7 units to the right, then a reflection over the *x*-axis
- A reflection over the *y*-axis, then a translation 8 units down

Common errors include confusing the *x*-axis with the *y*-axis; confusing clockwise and counterclockwise; confusing horizontal with vertical; and incorrectly locating the line $y = x$ or $y = -x$. *The answer is: a rotation 90° clockwise about the origin, then a reflection over the x-axis.*

Mathematical Connections: Students may benefit from additional practice finding the coordinates of vertices after a figure has been transformed.

Given: Triangle *ABC* with vertices located at *A*(1, 1), *B*(2, -3), and *C*(-1, -4).

Triangle *ABC* is reflected over the line $y = x$. What are the integral coordinates of point *C'* after this transformation?



Misconceptions sometimes occur when students do not plot the points first to see a visual representation of triangle ABC. Although this step is not required, it is helpful to solve problems if a picture is available. In this case, students must also understand what reflecting over or across the line $y = x$ means. What happens to the points during this transformation? Students may benefit from using graph paper to graph the pre-image of triangle ABC and the line $y = x$. Once they have done this, have them fold the graph paper on the line $y = x$. Using their pencil they can plot *A', B',* and *C'* by tracing over points *A, B,* and *C. The correct answer to this problem is C'* (-4, -1).

Concepts and Connections

CONCEPTS

Analyzing and describing geometric objects, the relationships and structures among them, or the space that they occupy can be used to classify, quantify, measure, or count one or more attributes.

CONNECTIONS

- *Within the grade level/course*:
  - G.TR.2 – The student will, given information in the form of a figure or statement, prove and justify two triangles are congruent using direct and indirect proofs, and solve problems involving measured attributes of congruent triangles.
  - G.TR.3 – The student will, given information in the form of a figure or statement, prove and justify two triangles are similar using direct and indirect proofs, and solve problems, including those in context, involving measured attributes of similar triangles.
  - G.PC.1 – The student will prove and justify theorems and properties of quadrilaterals and verify and use properties of quadrilaterals to solve problems, including the relationships between the sides, angles, and diagonals.
- *Vertical Progression*:
  - 8.MG.3 – The student will apply translations and reflections to polygons in the coordinate plane.
  - A.F.1 – The student will investigate, analyze, and compare linear functions algebraically and graphically, and model linear relationships.

Resources to Support Local Curriculum

- A list of approved textbooks and instructional materials is posted on the VDOE website.
- EOC Geometry Formula Sheet (PDF)

Triangles

Triangles comprise the foundation for trigonometric thinking, understanding, and application. Students explore triangle properties and formulas to develop a deeper understanding how to approach contextual situations. These skills unpack building blocks that are necessary for advanced mathematical thinking, understanding, and application. Also, mastery of applications involving triangles is required for higher level mathematics courses.

Throughout Geometry, students will determine the relationships between the measures of angles and lengths of sides in triangles, including contextual problems. Additionally, students will prove two triangles are congruent and solve contextual problems involving measured attributes of congruent triangles. Given a triangle, students will use geometric constructions to create a congruent triangle. Students will prove triangles are similar and solve contextual problems involving measured attributes of similar triangles. Also, students will solve problems, including contextual problems, involving trigonometry in right triangles and applications of the Pythagorean Theorem.

G.TR.1

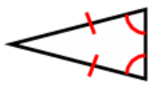
The student will determine the relationships between the measures of angles and lengths of sides in triangles, including problems in context.

Students will demonstrate the following Knowledge and Skills:

a) Given the lengths of three segments, determine whether a triangle could be formed.
b) Given the lengths of two sides of a triangle, determine the range in which the length of the third side must lie.
c) Order the sides of a triangle by their lengths when given information about the measures of the angles.
d) Order the angles of a triangle by their measures when given information about the lengths of the sides.
e) Solve for interior and exterior angles of a triangle, when given two angles.

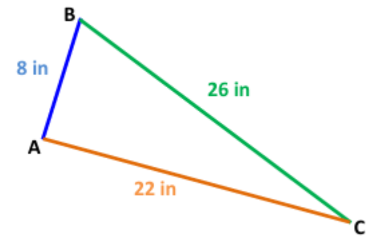
Understanding the Standard

- The longest side of a triangle is opposite the largest angle of the triangle and the shortest side is opposite the smallest angle.
- In a triangle, the lengths of two sides and the included angle determine the length of the side opposite the angle.
- Because isosceles triangles have two congruent sides, they also have two congruent angles.



- For a triangle to exist, the length of each side must be within a range that is determined by the lengths of the other two sides.

- Triangle Inequality Theorem: the sum of the lengths of any two sides of a triangle is greater than the length of the third side.

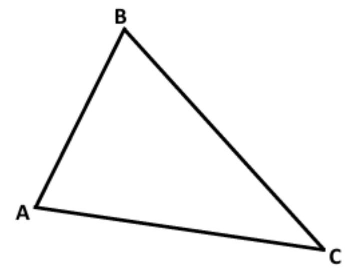


AB + BC > AC

AC + BC > AB

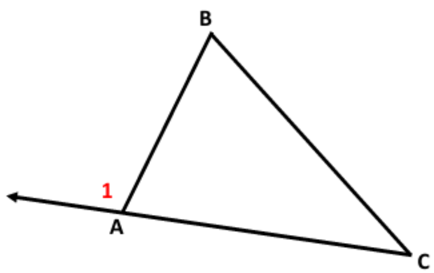
AB + AC > BC

- Triangle Angles Sum Theorem: the sum of the measures of the interior angles of a triangle is 180°.



$m\angle A + m\angle B + m\angle C = 180^\circ$

- Exterior Angle Theorem: an exterior angle of a triangle is equal to the sum of the two opposite interior angles.



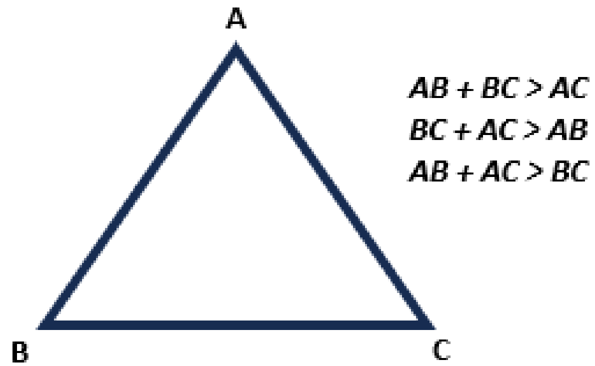
$m\angle 1 = m\angle B + m\angle C$

Skills in Practice

While the five process goals are expected to be embedded in each standard, the Skills in Practice highlight the most prevalent process goals in relation to the content presented.

Mathematical Problem Solving:

- Students must understand that the Triangle Inequality Theorem means the measure of any side of a triangle must be shorter than the measures of the other two sides added together. Refer to the diagram below.



- Sometimes students may not be given one of the side lengths of a triangle. They would benefit from opportunities to find all possible lengths for the missing side of a triangle when given the lengths of the other two sides.

  Given: Triangle *ABC* with *AB* = 42 and *BC* = 20

  Which of the following are possible lengths for *AC*?

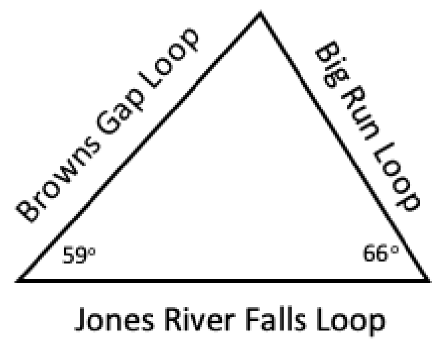
  12 20 22 32 42 50 62 70

  Students often struggle with the Triangle Inequality Theorem due to several misconceptions. A common error is applying it only to right triangles, confusing it with the Pythagorean Theorem. Another misconception is the belief that any three lengths can form a triangle, failing to understand that the sum of any two sides must be greater than the third. To help students gain a conceptual understanding of the triangle inequality theorem, teachers may provide straws, pipe cleaners, or wiki sticks of varying lengths to student groups. Students will use the straws to determine which combinations will and will not form a triangle. Have students discuss the relationships between the side measures of the combinations that formed triangles.

Teachers may also need to review the substitution property and illustrate how a side measure may be found by using substitution. In the example shown $AC > 42 - 20$ and $42 + 20 > AC$. Thus, $22 < AC < 62$. A quick way to find these values is to have students add and subtract the given two sides of the triangle. Have them write down those numbers. Tell students that the missing side falls between the sum and difference. AC cannot equal 22 or 62. This can be written as $AC \neq 22$ and $AC \neq 62$. The correct answer to this problem is {32, 42, 50}.

Mathematical Representations: Students will benefit from problems that are presented in a contextual format, when ordering and determining missing lengths and sides of triangles.

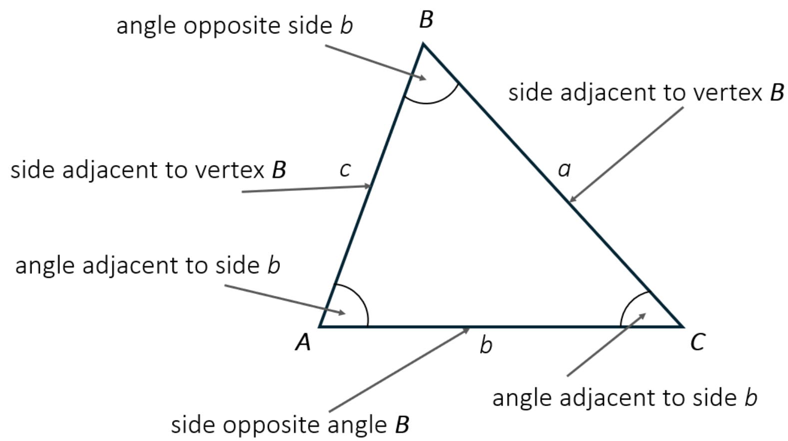
Shenandoah National Park has many hiking trails. In the south district location of the Shenandoah National Park, there are three hiking trails that are in close proximity to each other – Big Run Loop, Browns Gap Loop, and Jones River Falls Loop. These hiking trails create a triangle as shown. Given the angle measures of the triangle formed, determine the longest trail and the shortest trail. Place your response in the blanks provided below. The figure is not drawn to scale.



Longest trail: ________ Shortest trail: ________

A common misconception that students may have is thinking that not enough information is given to determine the longest and shortest trail. This may indicate that students do not recognize that the Triangle Sum Theorem must be applied to find the missing angle measure first before determining the longest and shortest sides of the triangle. Teachers are encouraged to emphasize with students to calculate the sum of the given angles; and then subtract the sum from 180° as the sum of the angles of any triangle is equivalent to 180°. Once the missing angle measure is known, teachers should model for students how to order the side opposite each angle from least to greatest.

Some students struggle with understanding where the “side opposite” an angle is. These students may benefit from a visual representation such as the figure shown.



Concepts and Connections

Concepts

Analyzing and describing geometric objects, the relationships and structures among them, or the space that they occupy can be used to classify, quantify, measure, or count one or more attributes.

Connections

- *Within the grade level/course*:
    - G.TR.2 **–** The student will, given information in the form of a figure or statement, prove and justify two triangles are congruent using direct and indirect proofs, and solve problems involving measured attributes of congruent triangles.
    - G.TR.3 **–** The student will, given information in the form of a figure or statement, prove and justify two triangles are similar using direct and indirect proofs, and solve problems, including those in context, involving measured attributes of similar triangles.
    - G.TR.4 **–** The student will model and solve problems, including those in context, involving trigonometry in right triangles and applications of the Pythagorean Theorem.
- *Vertical Progression*:
    - 7.MG.2 – The student will solve problems and justify relationships of similarity using proportional reasoning.
    - 8.MG.1 **–** The student will use the relationships among pairs of angles that are vertical angles, adjacent angles, supplementary angles, and complementary angles to determine the measure of unknown angles.

- 8.MG.4 **–** The student will apply the Pythagorean Theorem to solve problems involving right triangles, including those in context.

Resources to Support Local Curriculum

- A list of approved textbooks and instructional materials is posted on the VDOE website.
- EOC Geometry Formula Sheet (PDF)

G.TR.2

The student will, given information in the form of a figure or statement, prove and justify two triangles are congruent using direct and indirect proofs, and solve problems involving measured attributes of congruent triangles.

Students will demonstrate the following Knowledge and Skills:

a) Use definitions, postulates, and theorems (including Side-Side-Side (SSS); Side-Angle-Side (SAS); Angle-Side-Angle (ASA); Angle-Angle-Side (AAS); and Hypotenuse-Leg (HL)) to prove and justify two triangles are congruent.

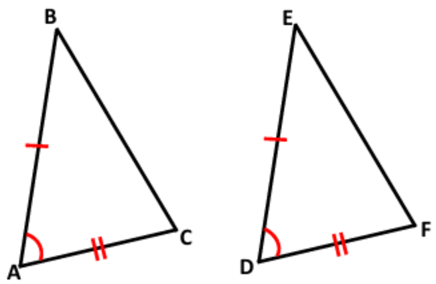
b) Use algebraic methods to prove that two triangles are congruent.

c) Use coordinate methods, such as the slope formula and the distance formula, to prove two triangles are congruent.

d) Given a triangle, use congruent segment, congruent angle, and/or perpendicular line constructions to create a congruent triangle (SSS, SAS, ASA, AAS, and HL).

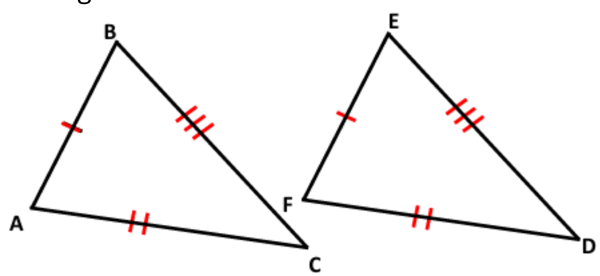
Understanding the Standard

- Physical tools to assist in student understanding the standard: physical protractor, ruler, compass, straight edge, and paper folding.
- Digital tools (dynamic software) in mathematical and drafting platforms may also assist in students' conceptual understanding.
- Deductive or inductive reasoning is used in mathematical proofs.
- Congruence does not depend on the position of the triangles.
- Congruent figures are similar, but similar figures are not necessarily congruent. This relationship can be depicted through a Venn diagram.
- Given coordinate representations of triangles, the distance, slope, and midpoint formulas may be used to prove triangles are congruent.
    - The distance between two points (x_1, y_1) and (x_2, y_2) is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
    - The slope of a line is the ratio of rise to run. If (x_1, y_1) and (x_2, y_2) are any two point son a line, the slope of the line is $m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$.
    - The midpoint M of $\overline{AB}$ with endpoints $A(x_1, y_1)$ and $B(x_2, y_2)$ is found by $M(\frac{x_{1+} x_2}{2}, \frac{y_{1+} y_2}{2})$.
-
- The phrase, *"Corresponding parts of congruent triangles are congruent,"* is abbreviated CPCTC.
- Two triangles can be proven congruent using the following criterion:
    - Side-Angle-Side (SAS): If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, then the triangles are congruent.



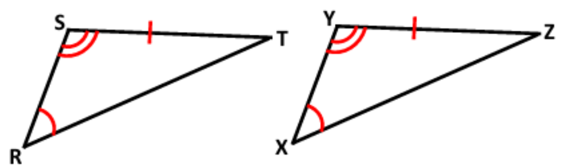
Example: If $\overline{AB} \cong \overline{DE}$, ∠A ≅ ∠D, and $\overline{AC} \cong \overline{DF}$, then ΔABC ≅ ΔDEF.

- Side-Side-Side (SSS): If all three sides of one triangle are congruent to the corresponding three sides of another triangle, then the two triangles are congruent.



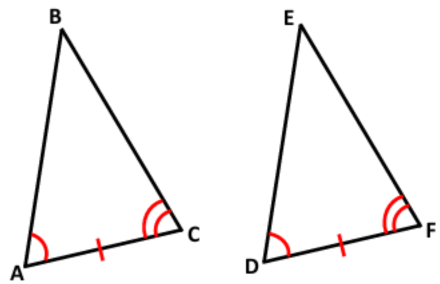
Example: If $\overline{AB} \cong \overline{FE}$, $\overline{AC} \cong \overline{FD}$, and $\overline{BC} \cong \overline{ED}$, then ΔABC ≅ ΔFED.

- Angle-Angle-Side (AAS): If two angles and a non-included side of one triangle are congruent to the corresponding two angles and non-included side of another triangle, then the two triangles are congruent.



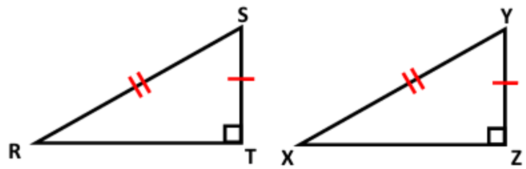
Example: If ∠R ≅ ∠X, ∠S ≅ ∠Y, and $\overline{ST} \cong \overline{YZ}$, then ΔRST ≅ ΔXYZ.

- Angle-Side-Angle (ASA): If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, then the two triangles are congruent.



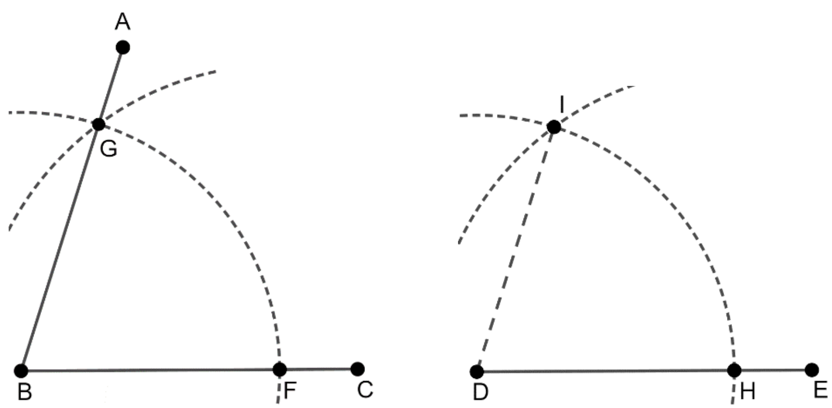
Example: If ∠A ≅ ∠D, $\overline{AC} \cong \overline{DF}$, and ∠C ≅ ∠F, then ΔABC ≅ ΔDEF.

- Hypotenuse-Leg (HL): If the hypotenuse and one leg of a right triangle are congruent to the corresponding hypotenuse and leg of another right triangle, then the two triangles are congruent.

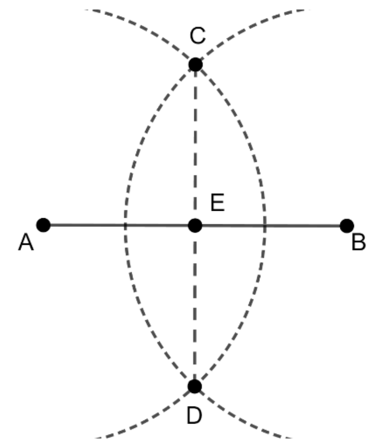


Example: If $\overline{RS} \cong \overline{XY}$, and $\overline{ST} \cong \overline{YZ}$, then ΔRST ≅ ΔXYZ.

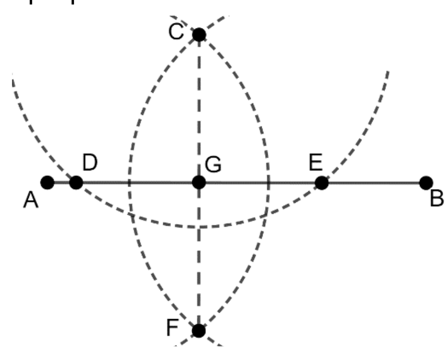
- Triangle congruency can be explored using geometric constructions.
- Given $\angle GBF$ and $\overline{DE}$, construct an angle congruent to $\angle GBF$. This creates $\overline{BF} \cong \overline{DH}$. The second arc creates $\overline{FG} \cong \overline{HI}$. Thus, $\Delta BFG \cong \Delta DHI$. Therefore $\angle GBF \cong \angle IDH$ by CPCTC.



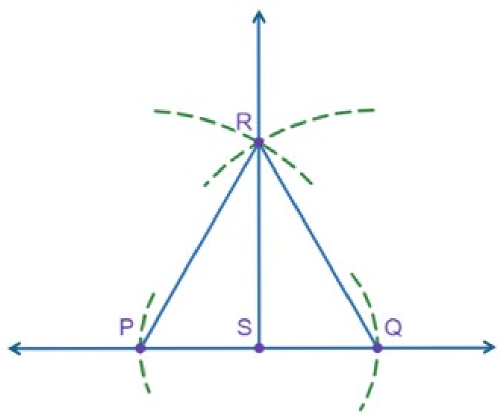
- In the example construction below, the construction of a perpendicular bisector creates a set of congruent triangles, $\Delta ACD \cong BCD$ by SSS. Using CPCTC, $\angle ACE \cong \angle BCE$. Also, because ΔACB is isosceles with $\overline{AC} \cong \overline{BC}$, $\Delta CAE \cong \Delta CBE$. Thus, $\Delta ACE \cong \Delta BCE$ by ASA. Because $\angle CEA \cong \angle CEB$, by CPCTC, and they form a linear pair, they must be right angles. This, combined with $\overline{AE} \cong \overline{BE}$ by CPCTC, proves that $\overline{CD}$ is the perpendicular bisector of $\overline{AB}$.



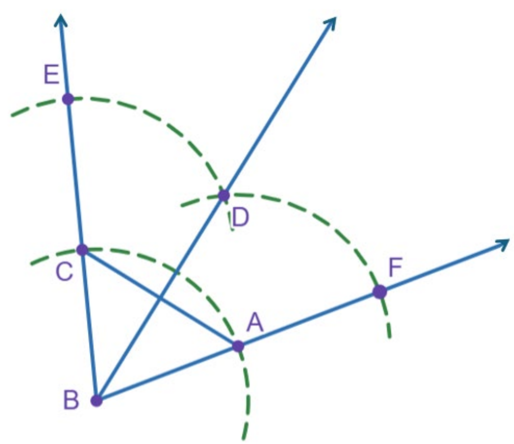
- The construction of the perpendicular to a given line from a point not on the line can be justified by proving $\Delta CDG \cong \Delta CEG$. The proof is very similar to the proof for the construction of the perpendicular bisector.



- The construction of the perpendicular to a given line from a point on the line can be justified using isosceles and congruent triangles by SSS.



- The construction of an angle bisector can be justified using congruent triangles by SSS.

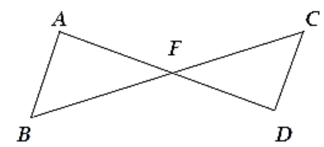


Skills in Practice

While the five process goals are expected to be embedded in each standard, the Skills in Practice highlight the most prevalent process goals in relation to the content presented.

Mathematical Communication: Students will benefit from additional practice completing the steps and reasons in two-column deductive proofs and paragraph proofs that prove triangles congruent.

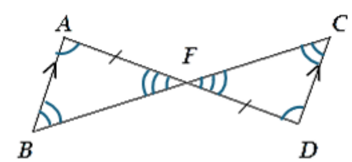
Given: $\overline{AB} \parallel \overline{CD}$, $\overline{AF} \cong \overline{FD}$



Prove: $\triangle ABF \cong \triangle DCF$

Misconceptions may occur when students incorrectly assume $\overline{AB} \cong \overline{CD}$. This means they think the parallel symbol also means congruence. Using the image above, students should identify everything they know about the triangles.

Given: $\overline{AB} \parallel \overline{CD}$, $\overline{AF} \cong \overline{FD}$



Prove: $\triangle ABF \cong \triangle DCF$

Some students will see the congruency markings and think $\Delta ABF \cong \Delta DCBF$ by AAA. If this happens, teachers may show triangles with varying side lengths but with the same angle measures. Students can use a ruler and protractor to see how AAA does not prove two triangles congruent. It may also be helpful for some students to separate the two triangles after they have marked them so they can visually see the relationship of ASA or AAS.

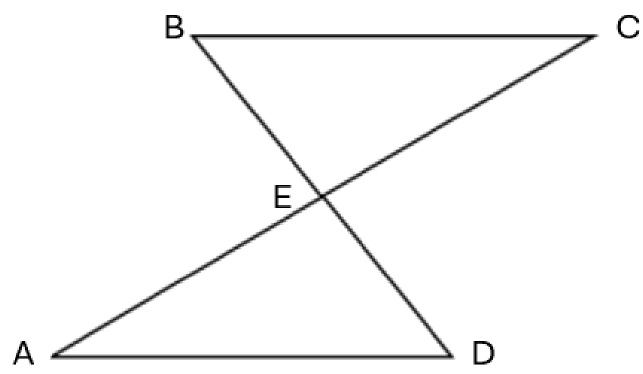
After triangles are marked, a congruence postulate must be used to prove the two triangles are congruent.

- $\angle C \cong \angle B$ because they are alternate interior angles.
- $\angle A \cong \angle D$ because they are alternate interior angles.
- $\angle AFB \cong \angle DFC$ because they are vertical angles.
- $\overline{AF} \cong \overline{FD}$ was given information.
- Using ASA or AAS $\Delta ABF \cong \Delta DCF$.

This information may now be used to create a formal two-column proof. Begin with the information that was given and provide the steps to prove the triangles congruent. This problem ahs more than one way to prove congruence. Have students share their approach and explain their methodology.

Mathematical Reasoning: Some students may struggle with two-column and paragraph proofs. It is important to reinforce the importance of drawing the figure and marking the given information. Encourage students to ask themselves "What does this given information tell me? What else do I know?" Engaging students in a discuss of what they do know will build confidence and knowledge of the process.

Given: $\overline{AC}$ and $\overline{BD}$ bisect each other at point *E*.



Prove: $\Delta\ AED \cong \Delta\ CEB$

Statements	Reasons
1. $\overline{AC}$ and $\overline{BD}$ bisect each other at point E.	1. Given
2. $\overline{AE} \cong \overline{CE}$ and $\overline{DE} \cong \overline{BE}$	2. Definition of segment bisector
3. $\overline{AE} \cong \overline{CE}$ and $\overline{DE} \cong \overline{BE}$	3. Definition of congruence
4. $\angle AED \cong \angle CEB$	4. Vertical angles are congruent.
5. $\Delta\, AED \cong \ \Delta\, CEB$	5. SAS (Side-Angle-Side) Theorem

Concepts and Connections

Concepts

Analyzing and describing geometric objects, the relationships and structures among them, or the space that they occupy can be used to classify, quantify, measure, or count one or more attributes.

Connections

- *Within the grade level/course*:
  - G.RLT.1 – The student will translate logic statements, identify conditional statements, and use and interpret Venn diagrams.
  - G.RLT.2– The student will analyze, prove, and justify the relationships of parallel lines cut by a transversal.
  - G.TR.1 – The student will determine the relationships between the measures of angles and lengths of sides in triangles, including problems in context.
  - G.TR.3 – The student will, given information in the form of a figure or statement, prove and justify two triangles are similar using direct and indirect proofs, and solve problems, including those in context, involving measured attributes of similar triangles.
  - G.TR.4– The student will model and solve problems, including those in context, involving trigonometry in right triangles and applications of the Pythagorean Theorem.
- *Vertical Progression*:
  - 6.MG.3 – The student will describe the characteristics of the coordinate plane and graph ordered pairs.
  - 6.MG.4 – The student will determine congruence of segments, angles, and polygons.
  - 7.MG.2 – The student will solve problems and justify relationships of similarity using proportional reasoning.

- 8.MG.4 **–** The student will apply the Pythagorean Theorem to solve problems involving right triangles, including those in context.
- A.F.1 **–** The student will investigate, analyze, and compare linear functions algebraically and graphically, and model linear relationships.

- *Digital Learning Integration:*
  - 9-12 ID.A. Students autonomously select and use appropriate technologies in a design process to generate ideas, create, document, test, revise, and present innovative products or solve authentic problems.

Resources to Support Local Curriculum

- A list of approved textbooks and instructional materials is posted on the VDOE website.
- EOC Geometry Formula Sheet (PDF)

G.TR.3

The student will, given information in the form of a figure or statement, prove and justify two triangles are similar using direct and indirect proofs, and solve problems, including those in context, involving measured attributes of similar triangles.

Students will demonstrate the following Knowledge and Skills:

a) Use definitions, postulates, and theorems (including Side-Angle-Side (SAS); Side-Side-Side (SSS); and Angle-Angle (AA)) to prove and justify that triangles are similar.

b) Use algebraic methods to prove that triangles are similar.

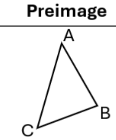
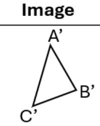
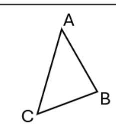
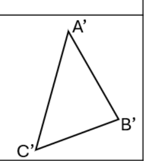
c) Use coordinate methods, such as the slope formula and the distance formula, to prove two triangles are similar.

d) Describe a sequence of transformations that can be used to verify similarity of triangles located in the same plane.

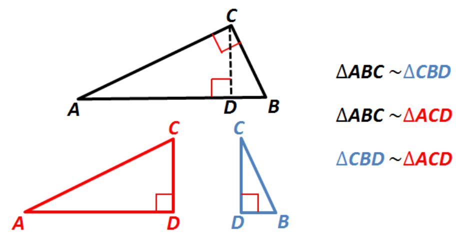
e) Solve problems, including those in context involving attributes of similar triangles.

Understanding the Standard

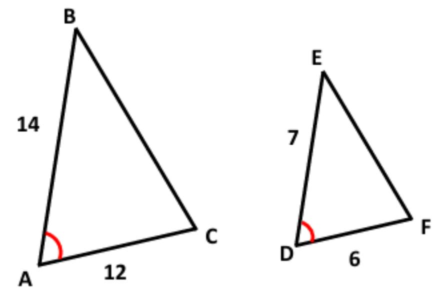
- Physical tools to assist in student understanding the standard: physical protractor, ruler, compass, straight edge, and paper folding.
- Digital tools (dynamic software) in mathematical and drafting platforms may also assist in students' conceptual understanding.
- Deductive or inductive reasoning is used in mathematical proofs.
- Similarity does not depend on the position of the triangles.
- Similar triangles are created using dilations.
- A dilation is a transformation in which an image is formed by enlarging or reducing the preimage proportionally by a scale factor from the center of dilation. The center of dilation may or may not be on the preimage.
    - The image of a dilation is similar to the preimage.
    - The scale factor determines the ratio of the image's size to the original triangle's size. If the scale factor is greater than 1, the image is larger; if it's between 0 and 1, the image is smaller.

	Preimage	Image
$0 < k < 1$		
$k > 1$		

- Congruent figures are also similar, but similar figures are not necessarily congruent. Congruence is a special case of similarity.
- Corresponding sides of similar triangles are proportional.
- Corresponding angles of similar triangles are congruent.
- Proportional reasoning is important when comparing attribute measures in similar figures.
- The altitude in a right triangle creates three similar right triangles.

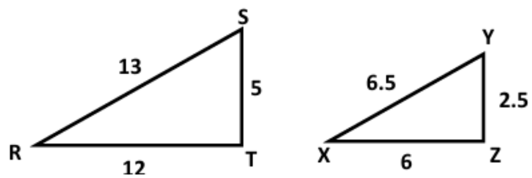


- Similar triangles are connected to triangles used in trigonometry, which is the study of relationships between angles and sides of triangles. Trigonometric functions are consistent among all right triangles.
- Side-Angle-Side (SAS) Triangle Similarity Theorem: If two sides of one triangle are proportional to two sides of another triangle and their included angles are congruent, then the triangles are similar.



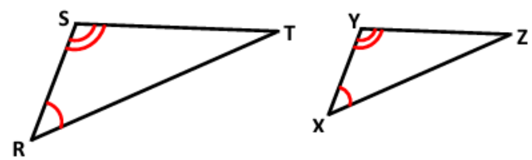
If $\angle A \cong \angle D$ and $\frac{AB}{DE} = \frac{AC}{DF}$ then $\Delta ABC \sim \Delta DEF$.

- Side-Side-Side (SSS) Triangle Similarity Theorem: If three sides of a triangle are proportional to the three corresponding sides of another triangle, then the triangles are similar.



If $\frac{RT}{XZ} = \frac{RS}{XY} = \frac{ST}{YZ}$ then ΔRST ~ ΔXYZ.

- Angle-Angle (AA) Triangle Similarity Postulate: If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.



If Angle ∠R ≅ ∠X and Angle ∠S ≅ ∠Y, then ΔRST ~ ΔXYZ.

Skills in Practice

While the five process goals are expected to be embedded in each standard, the Skills in Practice highlight the most prevalent process goals in relation to the content presented.

Mathematical Problem Solving: Students will benefit from additional practice proving triangles similar by using properties, postulates, or theorems.

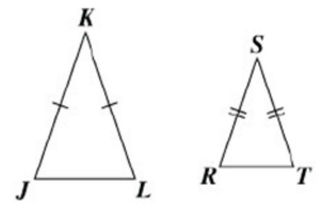
Complete the proof.

Given: $\triangle JKL$ and $\triangle RST$

$\angle JKL \cong \angle RST$

$\overline{JK} \cong \overline{KL}$ and $\overline{RS} \cong \overline{ST}$

Prove: $\triangle JKL \sim \triangle RST$



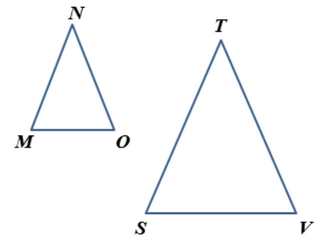
Statements	Reasons
1. $\angle JKL \cong \angle RST$	1. Given
2. $\overline{JK} \cong \overline{KL}$ and $\overline{RS} \cong \overline{ST}$	2. Given
3. $JK = KL$; $RS = ST$	3.
4. $\frac{JK}{RS} = \frac{KL}{ST}$	4.
5. $\triangle JKL \sim \triangle RST$	5.

Some students may not understand why “Division Property of Equality” is used for Reason 4. It may help to illustrate using numerical examples so that students may see the connection when dividing by equivalent values. It may also help to highlight $\overline{JK}$ and $\overline{RS}$ one color and $\overline{KL}$ and $\overline{ST}$ another color so students may see they are corresponding parts. The answers are:

3. Definition of congruence
4. Division property of equality
5. SAS Similarity

Mathematical Reasoning: Students need additional practice proving triangles similar when specific measurements are not given.

Refer to the image that follows. Given: Δ *MNO* and Δ *VTS*



Select two relationships that together would prove Δ*MNO* and Δ*VTS* by the Side-Angle-Side (SAS) Similarity Theorem.

- $\angle N \cong \angle T$
- $\angle M \cong \angle S$
- $\angle M \cong \angle V$
- $\frac{MN}{ST} = \frac{SV}{MO}$
- $\frac{MN}{VT} = \frac{MO}{VS}$

Some students may select $\angle N \cong \angle T$ and $\angle M \cong \angle S$. This indicates students did not understand they were to use the SAS Similarity Theorem. Other students may select the proportion $\frac{MN}{ST} = \frac{SV}{MO}$. These students incorrectly matched $\frac{\text{small triangle}}{\text{large triangle}} = \frac{\text{large triangle}}{\text{small triangle}}$. Various strategies such as highlighting the small triangle one color and the large triangle another color or placing a number 1 inside the small triangle and a number 2 inside the large triangle may help students track the order when writing proportions. *The answers are:* $\angle M \cong \angle V$ *and* $\frac{MN}{VT} = \frac{MO}{VS}$.

Concepts and Connections

Concepts

Analyzing and describing geometric objects, the relationships and structures among them, or the space that they occupy can be used to classify, quantify, measure, or count one or more attributes.

Connections

- *Within the grade level/course*:
  - G.RLT.1 – The student will translate logic statements, identify conditional statements, and use and interpret Venn diagrams.
  - G.TR.1 **–** The student will determine the relationships between the measures of angles and lengths of sides in triangles, including problems in context.
  - G.TR.2 **–** The student will, given information in the form of a figure or statement, prove and justify two triangles are congruent using direct and indirect proofs, and solve problems involving measured attributes of congruent triangles.
  - G.TR.4 **–** The student will model and solve problems, including those in context, involving trigonometry in right triangles and applications of the Pythagorean Theorem.
  - G.PC.1 – The student will prove and justify theorems and properties of quadrilaterals and verify and use properties of quadrilaterals to solve problems, including the relationships between the sides, angles, and diagonals.
  - G.DF.2 – The student will determine the effect of changing one or more dimensions of a three-dimensional geometric figure and describe the relationship between the original and changed figure.
- *Vertical Progression*:
  - 6.MG.3 – The student will describe the characteristics of the coordinate plane and graph ordered pairs.
  - 7.MG.2 – The student will solve problems and justify relationships of similarity using proportional reasoning.
  - 8.MG.4 **–** The student will apply the Pythagorean Theorem to solve problems involving right triangles, including those in context.
  - A.F.1 **–** The student will investigate, analyze, and compare linear functions algebraically and graphically, and model linear relationships.
- *Digital Learning Integration*:
  - 9-12 ID.A. Students autonomously select and use appropriate technologies in a design process to generate ideas, create, document, test, revise, and present innovative products or solve authentic problems.

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G.TR.4

The student will model and solve problems, including those in context, involving trigonometry in right triangles and applications of the Pythagorean Theorem.

Students will demonstrate the following Knowledge and Skills:

a) Determine whether a triangle formed with three given lengths is a right triangle.

b) Find and verify trigonometric ratios using right triangles.

c) Model and solve problems, including those in context, involving right triangle trigonometry (sine, cosine, and tangent ratios).

d) Solve problems using the properties of special right triangles.

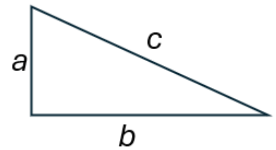
e) Solve for missing lengths in geometric figures, using properties of 45°-45°-90° triangles, where rationalizing denominators may be necessary.

f) Solve for missing lengths in geometric figures, using properties of 30°-60°-90° triangles, where rationalizing denominators may be necessary.

g) Solve problems, including those in context, involving right triangles using the Pythagorean Theorem and its converse, including recognizing Pythagorean Triples.

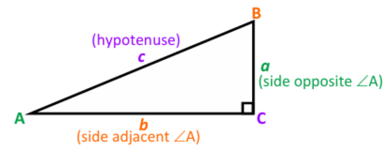
Understanding the Standard

- Physical tools to assist in student understanding the standard: physical protractor, ruler, compass, straight edge, and paper folding.
- Digital tools (dynamic software) in mathematical and drafting platforms may also assist in students' conceptual understanding.
- The converse of the Pythagorean Theorem can be used to determine if a triangle is a right triangle. If $c^2 = a^2 + b^2$, then the triangle is right.



- If a triangle is not a right triangle, the Pythagorean Inequality Theorem can be used to determine the type of triangle based on angles.
  - If $c^2 < a^2 + b^2$, then the triangle is acute.
  - If $c^2 > a^2 + b^2$, then the triangle is obtuse.
- Similar triangles can be used to develop the concept of Pythagorean triples and trigonometric ratios.

- The sine of an acute angle in a right triangle is equal to the cosine of its complement. This is because two acute angles are complementary. The side opposite one angle in a right triangle is adjacent to the other, making the sine and cosine ratios equal.
- Missing side lengths or angle measurements in a right triangle can be solved by using sine, cosine, and tangent ratios.

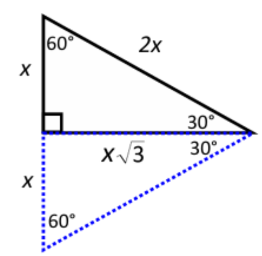


$$\sin A = \frac{\text{side opposite } \angle A}{\text{hypotenuse}} = \frac{a}{c}$$

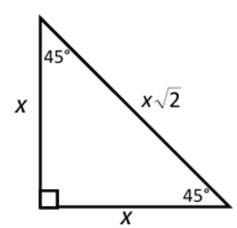
$$\cos A = \frac{\text{side adjacent } \angle A}{\text{hypotenuse}} = \frac{b}{c}$$

$$\tan A = \frac{\text{side opposite } \angle A}{\text{side adjacent to } \angle A} = \frac{a}{b}$$

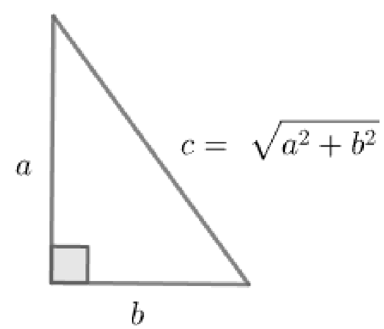
- 45°-45°-90° and 30°-60°-90° triangles are special right triangles because their side lengths can be specified as exact values using radicals rather than decimal approximations.
- 30°-60°-90° Triangle Theorem: In a 30°-60°-90° triangle, the length of the leg opposite the 60° angle is $\sqrt{3}$ times the length of the leg opposite the 30° angle. The hypotenuse is 2 times the length of the leg opposite the 30° angle.

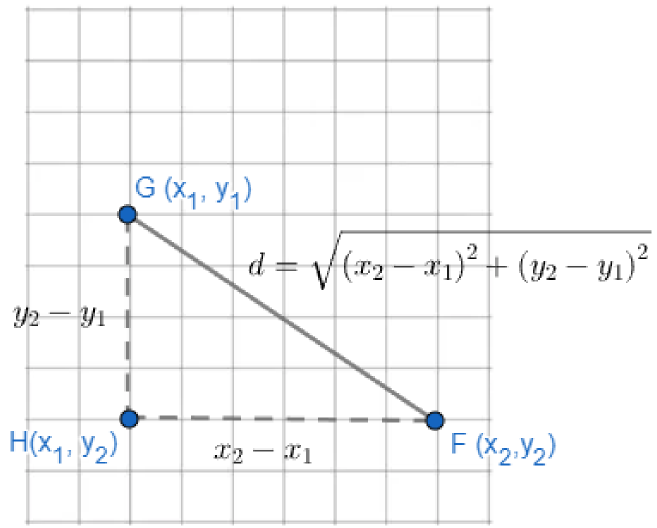


- 45°-45°-90° Triangle Theorem: In a 45°-45°-90° triangle, the hypotenuse is $\sqrt{2}$ times the length of a leg.



- A Pythagorean triple is a set of three positive integers that satisfy the equation $a^2 + b^2 = c^2$, where a and b, and c represent the side lengths of a right triangle. Examples include (3, 4, 5) and (5, 12, 13).
- Additional sets of Pythagorean triples can be found by applying properties for similar triangles and proportional sides. For example, doubling the sides of a triangle with side measures (in units) of (3, 4, 5) creates another Pythagorean triple of (6, 8, 10).
- The Pythagorean Theorem can be used to develop the distance formula. When the two endpoints are graphed, a right triangle can be drawn with the hypotenuse representing the diagonal distance between the points. The horizontal and vertical distance represent the legs of the right triangle and can be substituted for the variables a and b in the Pythagorean Theorem.
- The distance formula can be used to determine the length of a line segment when given the coordinates of the endpoints.





Skills in Practice

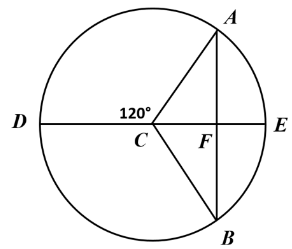
While the five process goals are expected to be embedded in each standard, the Skills in Practice highlight the most prevalent process goals in relation to the content presented.

Mathematical Connections:

- Students need practice applying properties of special right triangles to composite figures to determine unknown side lengths, particularly when radicals are used to represent the given information and/or the final answer.

Given: Circle *C* with diameter $\overline{DE}$

- Points *A* and *B* lie on circle *C* and form isosceles $\triangle ABC$
- $\overline{DE}$ bisects $\angle ACB$



If *DE* = 20 inches, what is *AF?*

A. 10 inches

B. 5 inches

C. $5\sqrt{3}$ inches

D. $10\sqrt{3}$ inches

Some students may believe *AF*=10, believing it to be half the length of the diameter. Students may struggle with recognizing $\overline{CA}$ is a radius of circle *C* and the hypotenuse of $\triangle CFA$. It may help students to outline the two right triangles formed by the median of the isosceles triangle. Encourage students to mark the angle measures of $\triangle CFA$ so they can visually see which is the long leg and the short leg of the 30-60-90 triangle. *The answer is: C. AF* = $5\sqrt{3}$ inches.

Students may need additional practice applying the Pythagorean Theorem when triangle side lengths are represented by algebraic expressions.

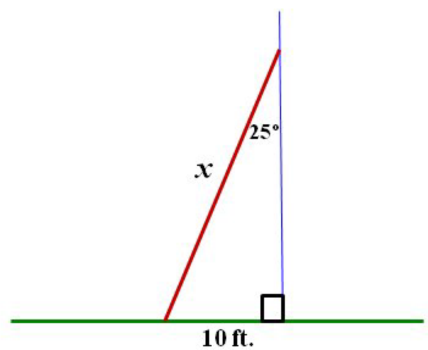
Triangle *ABC* has side lengths $25, 15x$ and $20x$. If the longest side is 25 units, what value of x proves $\triangle ABC$ is a right triangle?

A common misconception may include not correctly identifying the hypotenuse as 25 units. Other mistakes may include not squaring both the number and variable when it is used in the Pythagorean Theorem. Some students may incorrectly write $25^2 = 15x^2 + 20x^2$ and then combine like terms to get the equation $625 = 35x^2$. These students may benefit from a discussion on how $15x^2$ is different from $(15x)^2$. *The answer is:* $x = 1$.

Mathematical Representations: Students will benefit from additional practice using trigonometry to solve practical problems.

A ladder leans against a wall. The bottom of the ladder is 10 feet from the base of the wall, and the top of the ladder makes an angle of 25° with the wall. Find the length, x, of the ladder.

If an image is not provided in the practical problem, encourage students to draw a representation of the information provided. See figure below. Once they have drawn the figure ask, "What do we know about this triangle? What have we been given? Can we use the Pythagorean Theorem to solve for *x*? Why or why not? What can we use? Why?" Engaging students in mathematical discourse will strengthen their understanding of the right triangle concepts and their knowledge of knowing when to use a trigonometric function to solve a problem.



The answer is $x = 23.7$ *ft.*

A submarine sits on the surface of the ocean. The submarine dives at an angle of 15° and goes diagonally a distance of 900 meters before reaching the bottom. How deep is the water when the submarine reaches the bottom? What is the horizontal distance from where the submarine started to where it reached the bottom?

Students may struggle with creating a figure to represent this practical situation. Students may benefit from modeling the scenario with their hands to show the dive and how the submarine travels to towards the bottom of the ocean at an angle. Have students discuss in groups as a class how they visualize the figure and have them share their drawings with one another.
Some students will have difficulty since this problem has two unknown variables. They may not know where to begin. Have them focus on what they do know in the figure and address one unknown at a time. *The answer is: The water is 232.9m deep and the horizontal distance is 869.3m.*

Concepts and Connections

Concepts

Analyzing and describing geometric objects, the relationships and structures among them, or the space that they occupy can be used to classify, quantify, measure, or count one or more attributes.

Connections

- *Within the grade level/course*:
  - G.TR.1 **–** The student will determine the relationships between the measures of angles and lengths of sides in triangles, including problems in context.
  - G.TR.2 **–** The student will, given information in the form of a figure or statement, prove and justify two triangles are congruent using direct and indirect proofs, and solve problems involving measured attributes of congruent triangles.
  - G.TR.3 **–** The student will, given information in the form of a figure or statement, prove and justify two triangles are similar using direct and indirect proofs, and solve problems, including those in context, involving measured attributes of similar triangles.
  - G.PC.1 **–** The student will prove and justify theorems and properties of quadrilaterals and verify and use properties of quadrilaterals to solve problems, including the relationships between the sides, angles, and diagonals.
  - G.PC.2 – The student will verify relationships and solve problems involving the number of sides and measures of angles of convex polygons.
  - G.PC.3 – The student will solve problems, including those in context, by applying properties of circles.
- *Vertical Progression*:
  - 8.MG.4 – The student will apply the Pythagorean Theorem to solve problems involving right triangles, including those in context.
  - A.EI.1 – The student will represent, solve, explain, and interpret the solution to multistep linear equations and inequalities in one variable and literal equations for a specified variable.
  - A.EI.3 – The student will represent, solve, and interpret the solution to a quadratic equation in one variable.

Resources to Support Local Curriculum

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- EOC Geometry Formula Sheet (PDF)

Polygons and Circles

Polygons and circles are major components of geometrical thinking. Building upon the knowledge of triangles, polygons are expounded upon to foster critical thinking, understanding, and application of two- and three- dimensional figures. Students use applications of polygons and circles to develop their sense for contextual applications. These skills unpack building blocks that are necessary for advanced mathematical thinking, understanding, and application. Also, mastery of knowledge and skills relating to polygons and circles is required for higher level mathematics courses.

Throughout Geometry, students will prove and justify theorems and properties of quadrilaterals, and verify and use properties of quadrilaterals, including the relationships between the sides, angles, and diagonals, to solve problems, including those in context. Additionally, students will verify relationships and solve problems, including contextual problems, involving the number of sides and angles of convex polygons. Students will solve problems, including those in context, by applying properties of circles. Also, students will solve problems in the coordinate plane, including those in context, involving equations of circles.

G.PC.1

The student will prove and justify theorems and properties of quadrilaterals and verify and use properties of quadrilaterals to solve problems, including the relationships between the sides, angles, and diagonals.

Students will demonstrate the following Knowledge and Skills:

a) Solve problems, using the properties specific to parallelograms, rectangles, rhombi, squares, isosceles trapezoids, and trapezoids.

b) Prove and justify that quadrilaterals have specific properties, using coordinate and algebraic methods, such as the slope formula, the distance formula, and the midpoint formula.

c) Prove and justify theorems and properties of quadrilaterals using deductive reasoning.

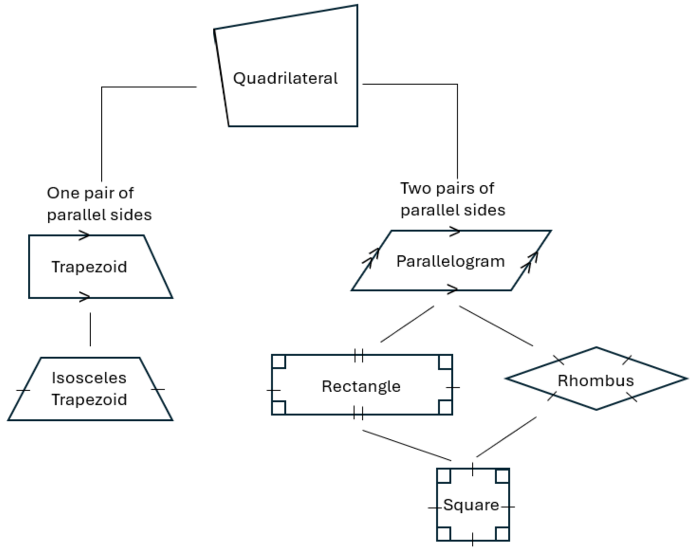
d) Use congruent segment, congruent angle, angle bisector, perpendicular line, and/or parallel line constructions to verify properties of quadrilaterals.

Understanding the Standard

- The properties of quadrilaterals can be verified experimentally using rulers, protractors, coordinate methods, and other measurement tools.
- A parallelogram is a quadrilateral with both pairs of opposite sides parallel. Additional properties of a parallelogram include:
  - opposite sides are congruent;

- opposite angles are congruent;
- consecutive angles are supplementary; and
- diagonals bisect each other.

- A rectangle is a parallelogram with four right angles. In addition to all the parallelogram properties, the properties of rectangles also include:
  - diagonals are congruent.
- A rhombus is a parallelogram with four congruent sides. In addition to all the parallelogram properties, the properties of rhombi also include:
  - all sides are congruent;
  - diagonals are perpendicular;
  - diagonals bisect opposite angles; and
  - diagonals divide the rhombus into four congruent right triangles.
- A square is a quadrilateral that is a regular polygon with four congruent sides and four right angles. In addition to all of the parallelogram, rhombus, and rectangle properties, the properties of squares also include:
  - diagonals divide the square into four congruent 45°-45°-90° triangles.
- To prove that a quadrilateral is a square, it must have four congruent sides and four right angles, or it must be shown it is both a rhombus and a rectangle.
- A trapezoid is a quadrilateral with exactly one pair of parallel sides. The parallel sides of a trapezoid are called bases. The nonparallel sides of a trapezoid are called legs.
- An isosceles trapezoid is a quadrilateral with one set of opposite sides parallel (bases) and the other set of opposite sides congruent (legs). In addition to all of the trapezoid properties, the properties of an isosceles trapezoid also include:
  - base angles are congruent; and
  - diagonals are congruent.
- The figure below illustrates several quadrilateral relationships.



- Properties of quadrilaterals can be used to identify a quadrilateral and to determine the measures of sides and angles.
- Given coordinate representations of quadrilaterals, the distance, slope, and midpoint formulas may be used to prove and justify that quadrilaterals have specific properties.
    - The distance between two points (x_1, y_1) and (x_2, y_2) is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
    - The slope of a line is the ratio of rise to run. If (x_1, y_1) and (x_2, y_2) are any two point son a line, the slope of the line is $m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$.
    - The midpoint M of $\overline{AB}$ with endpoints $A(x_1, y_1)$ and $B(x_2, y_2)$ is found by $M(\frac{x_{1+}\ x_2}{2}, \frac{y_{1+}\ y_2}{2})$.
- The angle relationships formed when parallel lines are intersected by a transversal can be used to prove the properties of quadrilaterals.
- Deductive reasoning can be used to prove and justify theorems of quadrilaterals. Examples include:
    - All rectangles have congruent diagonals. Quadrilateral ABCD is a rectangle. Quadrilateral ABCD has diagonals that are congruent.
    - If a quadrilateral is a rhombus, then its opposite sides are parallel. If a quadrilateral has opposite sides parallel, then it is a parallelogram. Conclusion: If a quadrilateral is a rhombus, then it is a parallelogram.
- Congruent triangles can be used to prove properties of quadrilaterals.

- The construction of the perpendicular bisector of a line segment can be justified using the perpendicular diagonals of a rhombus.
- The construction of the perpendicular to a given line from a point on, or not on, the line can be justified using the perpendicular diagonals of a rhombus.
- The construction of a bisector of a given angle can be justified using that the diagonals of a rhombus bisect the angles.
- Verify properties of quadrilaterals with compass constructions:

Property	**Construction**
Both pairs of opposite sides are parallel	Parallel line
Both pairs of opposite sides are congruent	Congruent segments
Both pairs of opposite angles are congruent	Congruent angles
Diagonals bisect each other	Congruent segments
One pair of opposite sides are congruent and parallel	Congruent segments and parallel lines
Diagonals are congruent	Congruent segments
All sides are congruent	Congruent segments
Diagonals are perpendicular bisectors of each other	Perpendicular line and congruent segments
Diagonals bisect opposite angles	Angle bisector
Nonparallel sides are congruent	Congruent segments
Base angles are congruent	Congruent angles

Skills in Practice

While the five process goals are expected to be embedded in each standard, the Skills in Practice highlight the most prevalent process goals in relation to the content presented.

Mathematical Problem Solving: Students need additional practice with contextual problems involving quadrilaterals.

Sasha built a window for her dollhouse in the shape of a quadrilateral. She knows the opposite sides of the window are parallel, but she wants to be sure it is in the shape of a rectangle. Which of these statements can she use to prove the window is a rectangle?

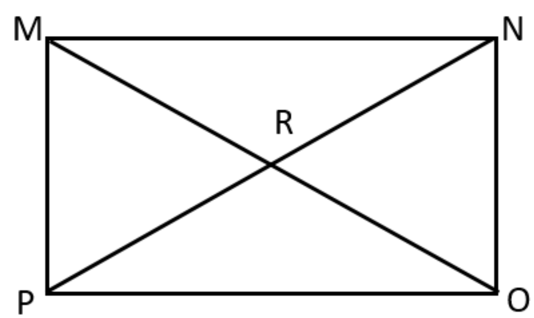
- The consecutive angles of the window are supplementary.
- The opposite sides of the window are congruent.

- The diagonals of the window bisect each other.
- The diagonals of the window are congruent.

Some students may believe the answer to be "The diagonals of the window bisect each other." Ask students to describe what distinguishes a rectangle from a parallelogram. Since a parallelogram has diagonals that bisect each other, this statement won't prove the window is a rectangle. Students may benefit from an activity where they are given various quadrilaterals and using a ruler and protractor, they must classify each quadrilateral. *The answer is: the diagonals of the window are congruent.*

Mathematical Reasoning:

- Rectangle MNOP is shown below with $m\angle \text{MON} = 62°$. Find $m\angle \text{OMN}$ and $m\angle \text{PRO}$. Explain your reasoning.



$m\angle \text{OMN} =$ _________

$m\angle \text{PRO} =$ _________

A common error a student may make is to find $m\angle \text{OMN} = 62°$. This may indicate the student views $\angle \text{MON}$ and $\angle \text{OMN}$ as opposite angles and concludes that they are congruent. Another common error a student may make would be to find $m\angle \text{RPO}$ instead of $m\angle \text{PRO}$. A student may struggle with identifying the correct angle when the angle is named by three letters instead of named by one letter or indicated by a number. To address these errors, teachers could encourage students to trace their angles to better visualize which angles are being represented in the problem. Students may also benefit from a discussion of why the four smaller triangles inside the rectangle are isosceles triangles. *The answer is:* $m\angle OMN = 28°$ *and* $m\angle PRO = 124°$.

Concepts and Connections

Concepts

Analyzing and describing geometric objects, the relationships and structures among them, or the space that they occupy can be used to classify, quantify, measure, or count one or more attributes.

Connections

- *Within the grade level/course*:
  - G.RLT.1 – The student will translate logic statements, identify conditional statements, and use and interpret Venn diagrams.
  - G.RLT.2 – The student will analyze, prove, and justify the relationships of parallel lines cut by a transversal.
  - G.TR.1 – The student will determine the relationships between the measures of angles and lengths of sides in triangles, including problems in context.
  - G.TR.4 – The student will model and solve problems, including those in context, involving trigonometry in right triangles and applications of the Pythagorean Theorem.
  - G.PC.2 – The student will verify relationships and solve problems involving the number of sides and measures of angles of convex polygons.
- *Vertical Progression*:
  - 6.MG.3 – The student will describe the characteristics of the coordinate plane and graph ordered pairs.
  - 7.MG.3 – The student will compare and contrast quadrilaterals based on their properties and determine unknown side lengths and angle measures of quadrilaterals.
  - 8.MG.1 – The student will use the relationships among pairs of angles that are vertical angles, adjacent angles, supplementary angles, and complementary angles to determine the measure of unknown angles.
  - 8.MG.5 – The student will solve area and perimeter problems involving composite plane figures, including those in context.
  - A.F.1 – The student will investigate, analyze, and compare linear functions algebraically and graphically, and model linear relationships.
- *Digital Learning Integration*:
  - 9-12 ID.A. Students autonomously select and use appropriate technologies in a design process to generate ideas, create, document, test, revise, and present innovative products or solve authentic problems.

Resources to Support Local Curriculum

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- EOC Geometry Formula Sheet (PDF)
- Quadrilateral Construction Exemplar Mathematics Instructional Plan (Word | PDF)

G.PC.2

The student will verify relationships and solve problems involving the number of sides and measures of angles of convex polygons.

Students will demonstrate the following Knowledge and Skills:

a) Solve problems involving the number of sides of a regular polygon given the measures of the interior and exterior angles of the polygon.

b) Justify the relationship between the sum of the measures of the interior and exterior angles of a convex polygon and solve problems involving the sum of the measures of the angles.

c) Justify the relationship between the measure of each interior and exterior angle of a regular polygon and solve problems involving the measures of the angles.

Understanding the Standard

- In convex polygons, each interior angle has a measure less than 180°.
- In concave polygons, one or more interior angles have a measure greater than 180°.
- A regular polygon is a convex polygon that is both equiangular (all angles congruent) and equilateral (all sides congruent).
- The sum of the measures of the interior angles of a convex polygon may be found by dividing the interior of the polygon into nonoverlapping triangles and multiplying the number of triangles created by 180°.
- An exterior angle is formed by extending a side of a polygon.
- The exterior angle and the corresponding interior angle of a polygon form a linear pair.
- The sum of exterior angles in any convex polygon is 360°.
- As the number of sides increases in a regular polygon, the measure of each interior angle increases and the measure of each exterior angle decreases.
- Given a number of sides, *n*, the following chart can be used to organize the polygon formulas.

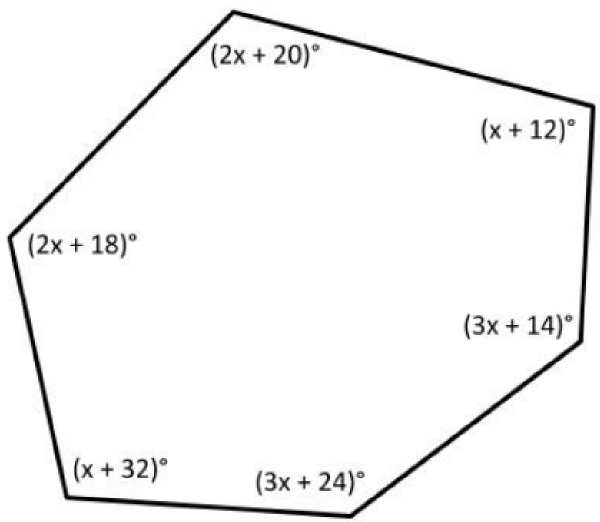
	Interior	Exterior
Sum of Angle Measures	$(n-2)180°$	$360°$
Angle Measures of Regular Polygon	$\frac{(n-2)180°}{n}$	$\frac{360°}{n}$

Skills in Practice

While the five process goals are expected to be embedded in each standard, the Skills in Practice highlight the most prevalent process goals in relation to the content presented.

Mathematical Problem Solving: Students will benefit from examples finding the measure of angles of convex polygons.

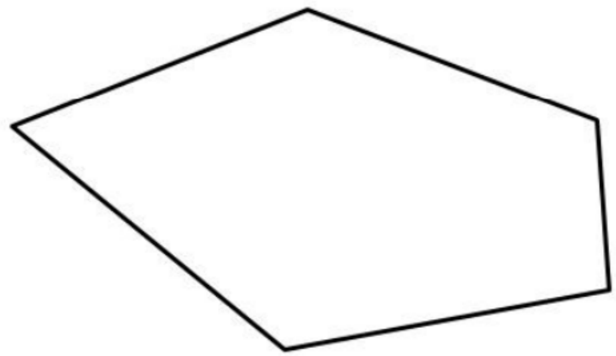
- A convex polygon is shown. What is the value of x?



A common error a student may make is to write an equation where the sum of the measures of the interior angles is set equal to 360°. This may indicate that a student has confused the sum of the exterior angles with the sum of the interior angles of a convex hexagon. Students may benefit from analyzing the interior angle sum as a pattern that increases by 180 with each added side.

Mathematical Communication: Students will benefit from multiple representations used to solve problems involving convex polygons.

Kelvin would like to find the sum of the interior angles in a pentagon. Kelvin thinks that if he can divide the pentagon into triangles, he can find the total interior angle sum.



a) How many non-overlapping triangles can be formed by drawing all possible diagonals from one vertex of a pentagon? Use the diagram to draw them.

A common misconception that some students have is assuming that the number of triangles will be equivalent to the number of sides. This misconception indicates that students will likely misuse or incorrectly remember the formula for the sum of the interior angles as $180°n$ instead of $180°(n - 2)$. Students may benefit from determining the number of triangles as part of a pattern that increases by 1 with every side length that is added.

b) Use what you know about the angle measures in each triangle to find the sum of the interior angles of a pentagon.

A common error is that even after determining the number of triangles correctly, a student may still apply the interior angle sum theorem incorrectly by using either $180°n$ or $360°(n - 2)$. This error indicates that the student does not understand and connect to the conceptual basis of the interior angle sum theorem. One strategy would be to have students work through a small subset of polygons, draw in the non-overlapping triangles from one vertex and use the triangle angle sum in order to generalize how to find the sum of the interior angles of a convex polygon.

Concepts and Connections

CONCEPTS

Analyzing and describing geometric objects, the relationships and structures among them, or the space that they occupy can be used to classify, quantify, measure, or count one or more attributes.

CONNECTIONS

- *Within the grade level/course*:
  - G.TR.1 – The student will determine the relationships between the measures of angles and lengths of sides in triangles, including problems in context.
  - G.PC.1 – The student will prove and justify theorems and properties of quadrilaterals and verify and use properties of quadrilaterals to solve problems, including the relationships between the sides, angles, and diagonals.
- *Vertical Progression*:
  - 7.MG.3 – The student will compare and contrast quadrilaterals based on their properties and determine unknown side lengths and angle measures of quadrilaterals.
  - 8.MG.1 **–** The student will use the relationships among pairs of angles that are vertical angles, adjacent angles, supplementary angles, and complementary angles to determine the measure of unknown angles.

Resources to Support Local Curriculum

- A list of approved textbooks and instructional materials is posted on the VDOE website.
- EOC Geometry Formula Sheet (PDF)
- Angles in Polygons Exemplar Mathematics Instructional Plan (Word | PDF)

G.PC.3

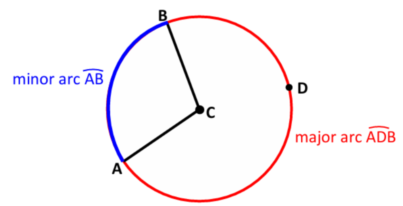
The student will solve problems, including those in context, by applying properties of circles.

Students will demonstrate the following Knowledge and Skills:

a) Determine the proportional relationship between the arc length or area of a sector and other parts of a circle.
b) Solve for arc measures and angles in a circle formed by central angles.
c) Solve for arc measures and angles in a circle involving inscribed angles.
d) Calculate the length of an arc of a circle.
e) Calculate the area of a sector of a circle.
f) Apply arc length or sector area to solve for an unknown measurement of the circle including the radius, diameter, arc measure, central angle, arc length, or sector area.

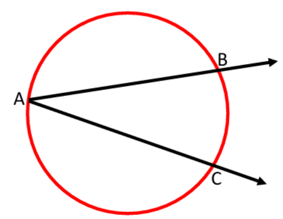
Understanding the Standard

- A central angle is an angle whose vertex is the center of the circle.



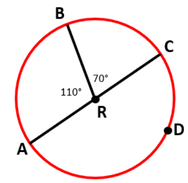
$\angle ACB$ is a central angle of circle C.

- An inscribed angle is an angle whose vertex is a point on the circle and whose sides contain chords of the circle.



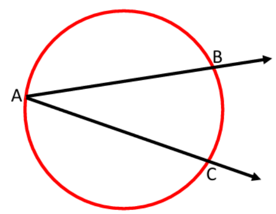
$\angle BAC$ is an inscribed angle.

- The measure of a central angle is equal to the measure of its intercepted arc.



Minor arcs	Major arcs	Semicircles
$m\overset{\frown}{AB} = 110°$	$m\overset{\frown}{BDA} = 250°$	$m\overset{\frown}{ADC} = 180°$
$m\overset{\frown}{BC} = 70°$	$m\overset{\frown}{BAC} = 290°$	$m\overset{\frown}{ABC} = 180°$

- The measure of an inscribed angle is half the measure of its intercepted arc.



$$m\angle BAC = \frac{1}{2} m\widehat{BC}$$

- Portions of circles can be thought of in three ways:
    - Arc Measure: Measured in degrees or radians and expresses the portion of the circle out of the whole. It is not affected by the size of the circle. Radian is not introduced as a unit of measure until Trigonometry.
    - Arc Length: Measured in linear units (e.g., inches, meters) and expresses the length of a particular arch. It is affected by the size of the circle.
    - Sector Area: Measured in square units (e.g., $in.^2$, m^2) and expresses how much area is contained within the sector. It is affected by the size of the circle.
- The ratio of the central angle to 360° is proportional to the ratio of the arc length to the circumference of the circle.
- The ratio of the central angle to 360° is proportional to the ratio of the area of the sector to the area of the circle.

Skills in Practice

While the five process goals are expected to be embedded in each standard, the Skills in Practice highlight the most prevalent process goals in relation to the content presented.

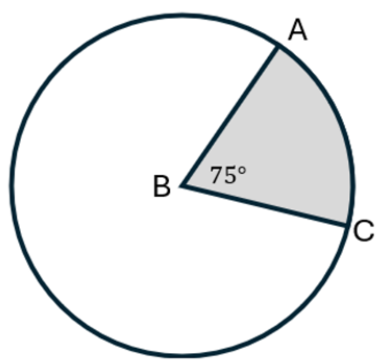
Mathematical Problem Solving: Students will benefit from using problem-solving strategies to unpack and solve problems given in context regarding circles.

- Bob divides his circular garden into 10 congruent sectors to plant different types of flowers. The circumference of Bob's garden is 50.5 feet.
  What is the area of one sector of Bob's garden?

  Students may have difficulty knowing where to begin when presenting with a practical problem. Encourage students to always begin with a figure or diagram. Follow-up with asking students to determine what they know from the given information and what they can use that information to find out. In this example, students know the circumference of Bob's garden. When the circumference is known, students can then determine the diameter and/or radius. Next, remind students to look back at the problem to determine what is being asked – what is the area of one sector of Bob's garden. Keeping students focused on drawing a diagram, determining what they know from given information, and being mindful of what question is being asked will help them decide what formulas are applicable to the situation. *The answer is approximately 20.3 sq. ft.*

Mathematical Reasoning: Students will benefit from additional practice using a measure of one part of the circle to find measures of other parts of the circle.

The radius of circle *B* is 4 inches. Which proportion could be used to determine the area of the shaded region?



$$\frac{75°}{360°} = \frac{x}{8\pi}$$

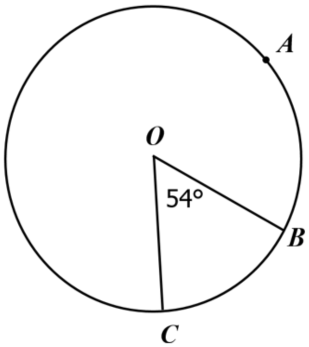
$$\frac{75°}{360°} = \frac{x}{4\pi}$$

$$\frac{75°}{360°} = \frac{x}{16\pi}$$

$$\frac{75°}{360°} = \frac{x}{8\pi}$$

Common misconceptions exist when students do not appear to recognize that the ratio of the central angle to the whole circle is equal to the ratio of the area of the sector to the area of the whole circle. *The answer is:* $\frac{75°}{360°} = \frac{x}{16\pi}$.

In Circle *O*, the length of $\widehat{BC}$ is 25.4 cm.
Find the length of $\widehat{BAC}$ to the nearest tenth of a centimeter.



One common misconception is when students do not appear to recognize that the ratio of the central angle to the whole circle is equal to the ratio of the arc length to the circumference. Some students may approach this problem using the formula for arc length which is also an appropriate way to solve the problem. Students would benefit from a discussion on the various approaches and how they derived an answer. *The answer is: the length of* $\widehat{BAC}$ *=143.9 cm.*

Concepts and Connections

CONCEPTS

Analyzing and describing geometric objects, the relationships and structures among them, or the space that they occupy can be used to classify, quantify, measure, or count one or more attributes.

CONNECTIONS

- *Within the grade level/course*:
  - G.PC.4 – The student will solve problems in the coordinate plane involving equations of circles.
- *Vertical Progression*:
  - 6.MG.1 – The student will identify the characteristics of circles and solve problems, including those in context, involving circumference and area.

- 8.MG.5 – The student will solve area and perimeter problems involving composite plane figures, including those in context.
- A.EI.1 – The student will represent, solve, explain, and interpret the solution to multistep linear equations and inequalities in one variable and literal equations for a specified variable.
- A.EI.3 – The student will represent, solve, and interpret the solution to a quadratic equation in one variable.

Resources to Support Local Curriculum

- A list of approved textbooks and instructional materials is posted on the VDOE website.
- EOC Geometry Formula Sheet (PDF)

G.PC.4

The student will solve problems in the coordinate plane involving equations of circles.

Students will demonstrate the following Knowledge and Skills:

a) Derive the equation of a circle given the center and radius using the Pythagorean Theorem.

b) Solve problems in the coordinate plane involving equations of circles:

   i) given a graph or the equation of a circle in standard form, identify the coordinates of the center of the circle;

   ii) given the coordinates of the endpoints of a diameter of a circle, determine the coordinates of the center of the circle.

   iii) given a graph or the equation of a circle in standard form, identify the length of the radius or diameter of the circle.

   iv) given the coordinates of the endpoints of the diameter of a circle, determine the length of the radius or diameter of the circle.

   v) given the coordinates of the center and the coordinates of a point on the circle, determine the length of the radius or diameter of the circle; and

   vi) given the coordinates of the center and length of the radius of a circle, identify the coordinates of a point(s) on the circle.

c) Determine the equation of a circle given:

   i) a graph of a circle with a center with coordinates that are integers;

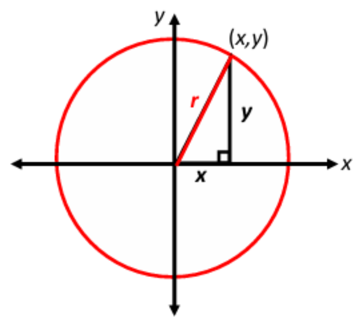
   ii) coordinates of the center and a point on the circle;

   iii) coordinates of the center and the length of the radius or diameter; and

   iv) coordinates of the endpoints of a diameter.

Understanding the Standard

- A circle is a locus of points equidistant from a given point, the center.
- The distance between any point on the circle and the center is the length of the radius.
- The equation of a circle with a given center and radius can be derived using the Pythagorean Theorem as follows: Every point (x, y) on a circle is the same distance from the center of the circle (h, k). This distance is defined as the radius (r). To determine the distance (radius) between (x, y) and (h, k), a right triangle is created with the hypotenuse as the radius (r) and the legs defined as $(x – h)$ and $(y – k)$. Substituting these variables into the Pythagorean Theorem results in the following equation: $(x – h)^2 + (y – k)^2 = r^2$.



- Standard form for the equation of a circle is $(x - h)^2 + (y - k)^2 = r^2$, where the coordinates of the center of the circle are (*h, k*) and *r* is the length of the radius.
- The midpoint of the diameter is the center of the circle.
- The midpoint formula can be used to find the center of the circle when given two endpoints.
- The distance formula can be used to determine the length of a radius given an endpoint and a midpoint.
- The equation of a circle gives the coordinates of every point, (*x, y*), on the circle.

Skills in Practice

While the five process goals are expected to be embedded in each standard, the Skills in Practice highlight the most prevalent process goals in relation to the content presented.

Mathematical Problem Solving: Students need practice using the equation of a circle to identify the radius, diameter, center, and/or a point on the circle.

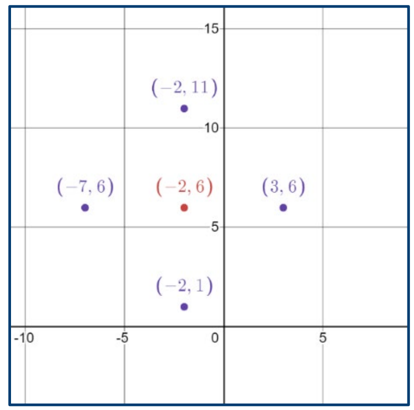
- Circle *O* has a center located at (-2, 6). The diameter is 10 units.
    a. What is the equation of the Circle *O*?
    b. Determine two integral coordinates that lie on Circle *O*.

  When writing the equation of a circle some students will write $(x-2)^2 + (y+6)^2 = 100$. This indicates they do not have an understanding of the circle formula and the signs of the center point (h, k). It also indicates they did not find the radius from the given diameter. Students would benefit from writing the circle formula and substituting the values of the center point using parenthesis to show the substitution: $(x-(-2))^2 + (y-(6))^2 = (5)^2$. To find integral coordinates that lie on the circle, students need opportunities to graph using graph paper and a compass. This will help them to understand the relationship between the center of the circle and the radius. Once you have students draw a circle using the center and radius, ask them to locate integral coordinates on the circle. Follow up with asking students to describe how these points are related to the center. Is there a way to determine the integral coordinates on the circle without graphing? Engaging students in this mathematical discussion will help them develop a conceptual understanding of how you can add or subtract the radius from the x- and y-values of the center point to find points that lie on a circle. *Answers to this problem are shown below. Graphical representations are provided.*

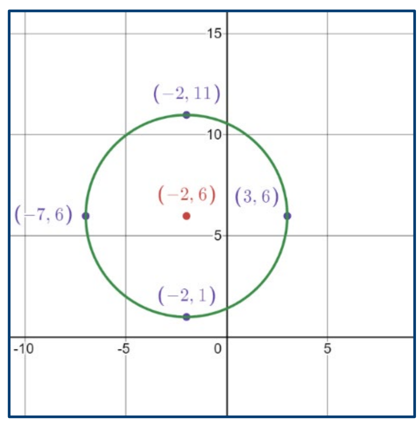
a. $(x + 2)^2 + (y - 6)^2 = 25$

b. Possible points: (-2, 1), (-2, 11), (-7, 6), (3, 6)

Graph the center and four points found by moving horizontally and vertically on the coordinate plane in r units of distance.



Graph the circle's equation to confirm that the four points lie on the circle.



Mathematical Reasoning: Students will benefit from practice identifying the center, radius, and diameter of a circle, given the equation written in standard form.

The equation of a circle is $(x - 3)^2 + (y + 4)^2 = 16$.

a. What are the coordinates of the center of the circle?

b. What is the radius of the circle?

c. What is the diameter of the circle?

d. Give the integral coordinates of two points that lie on the circle.

Errors might occur when students forget to apply the negative sign when addition is reflected in the equation. Also, students may forget to take the square root of the constant to find the radius. *Answers to this problem are shown below:*

a. (3, -4)

b. 4

c. 8

d. Possible points: (-1,-4) (7,-4) (3,-8) (3,0)

Concepts and Connections

CONCEPTS

Analyzing and describing geometric objects, the relationships and structures among them, or the space that they occupy can be used to classify, quantify, measure, or count one or more attributes.

CONNECTIONS

- *Within the grade level/course*:
  - G.TR.4 **–** The student will model and solve problems, including those in context, involving trigonometry in right triangles and applications of the Pythagorean Theorem.
  - G.PC.3 – The student will solve problems, including those in context, by applying properties of circles.
- *Vertical Progression*:
  - 6.MG.1 – The student will identify the characteristics of circles and solve problems, including those in context, involving circumference and area.

    - 8.MG.4 **–** The student will apply the Pythagorean Theorem to solve problems involving right triangles, including those in context.
- *Digital Learning Integration*:
    - 9-12 ID.A. Students autonomously select and use appropriate technologies in a design process to generate ideas, create, document, test, revise, and present innovative products or solve authentic problems.

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Two- and Three-Dimensional Figures

Two- and three-dimensional figures comprise the foundation for study, understanding, and application of geometry. Students use models and graphs to develop necessary skills used to solve contextual problems regarding two- and three-dimensional figures. These skills unpack building blocks that are required for advanced mathematical thinking, understanding, and application throughout physical science courses, engineering, and calculus.

Throughout Geometry, students will create models and solve problems, including those in context, involving surface area and volume of three-dimensional objects. Additionally, students will determine the effects of changing one or more dimensions of a three-dimensional figure, including recognizing when two- and three-dimensional figures are similar.

G.DF.1

The student will create models and solve problems, including those in context, involving surface area and volume of rectangular and triangular prisms, cylinders, cones, pyramids, and spheres.

Students will demonstrate the following Knowledge and Skills:

a) Identify the shape of a two-dimensional cross-section of a three-dimensional figure.

b) Create models and solve problems, including those in context, involving surface area of three-dimensional figures, as well as composite three-dimensional figures.

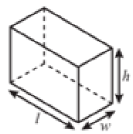
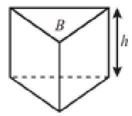
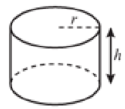
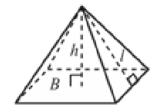
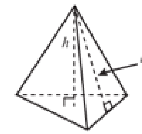
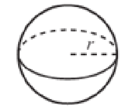
c) Solve multistep problems, including those in context, involving volume of three-dimensional figures, as well as composite three-dimensional figures.

d) Determine unknown measurements of three-dimensional figures using information such as length of a side, area of a face, or volume.

Understanding the Standard

- A cylinder is a solid figure formed by two congruent parallel faces called bases joined by a curved surface. In this course, cylinders are limited to right circular cylinders.
- A prism is a polyhedron that has a congruent pair of parallel bases and faces that are parallelograms. In this course, prisms are limited to right prisms.
- A pyramid is a polyhedron with a base that is a polygon and three or more faces that are triangles with a common vertex.
- Pyramids are limited to right regular pyramids with triangular, square, or hexagonal bases.
- A cone is a solid figure formed by a face called a base that is joined to a vertex (apex) by a curved surface. In this course, cones are limited to right circular cones.

- Slicing a three-dimensional figure using a geometric plane results in an intersection that is a two-dimensional figure. Slicing vertically, horizontally, or diagonally can result in cross-sections that include circles, triangles, rectangles, squares, ellipses, parabolas, and trapezoids.
- Subdivision of polygons may assist in determining the area of regular polygons.
- The formula for the area of regular polygon is $A = \frac{1}{2}ap$, where:
  - a represents apothem which is the perpendicular distance from any side to the center of the regular polygon; and
  - p represents the length of the polygon's perimeter.
- Generalize the surface area formula of prisms and cylinders to $SA = LA + 2B$.
- Generalize the surface area formula of cones and pyramids to $SA = LA + B$.
- The surface area of a prism or pyramid is the sum of the areas of all its faces.
- The surface area of a cylinder, cone, or hemisphere is the sum of the areas of the curved surface and base(s).
- The surface area of a sphere is the area of the curved surface.
- Surface area of spheres, cones, and cylinders should be considered in terms of pi or as a decimal approximation.
- Calculators may be used to determine decimal approximations for results.
- The lateral area (*LA*) of a cylinder or a cone is the area of the curved surface of the cylinder or cone, not including the base(s).
- The lateral area (*LA*) of a prism or a pyramid is the sum of the areas of all faces, not including the base(s).
- Generalize the volume formula of prisms and cylinders to $V = Bh$.
- Generalize the volume formula of cones and pyramids to $V = \frac{1}{3}Bh$.
- Volume and surface area of spheres, cones and cylinders should be considered in terms of pi or as a decimal approximation.

Right Rectangular Prism	Right Triangular Prism	Right Circular Cylinder	Right Circular Cone	Right Square-based Pyramid	Right Triangular Pyramid	Sphere
 $V = lwh$ $S.A. = 2lw + 2lh + 2wh$	 $V = Bh$ $L.A. = hp$ $S.A. = hp + 2B$	 $V = \pi r^2 h$ $L.A. = 2\pi rh$ $S.A. = 2\pi r^2 + 2\pi rh$	 $V = \frac{1}{3}Bh$ $V = \frac{1}{3}\pi r^2 h$ $L.A. = \pi rl$ $S.A. = \pi r^2 + \pi rl$	 $V = \frac{1}{3}Bh$ $L.A. = \frac{1}{2}lp$ $S.A. = \frac{1}{2}lp + B$	 $V = \frac{1}{3}Bh$ $L.A. = \frac{1}{2}lp$ $S.A. = \frac{1}{2}lp + B$	 $V = \frac{4}{3}\pi r^3$ $S.A. = 4\pi r^2$

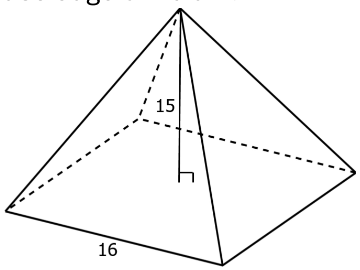
- Composite figures consist of two or more three-dimensional figures. The surface area of a composite figure may not be equal to the sum of the surface areas of the individual figures.
- Composite figures consist of two or more three-dimensional figures. The volume of a composite figure equals the sum or difference of the volumes of the individual figures.

Skills in Practice

While the five process goals are expected to be embedded in each standard, the Skills in Practice highlight the most prevalent process goals in relation to the content presented.

Mathematical Problem Solving: Students need practice finding surface area and/or volume of three-dimensional figures when some values are not directly provided on the figure and/or when the answer is expressed in terms of pi (π).

- A square pyramid has a height of 15 cm and a base edge of 16 cm.

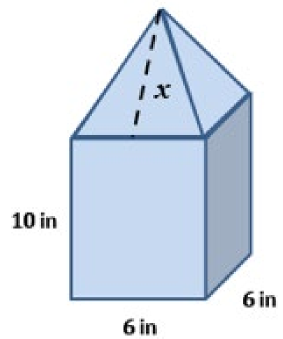


What is the area of one face of this pyramid?

Students often confuse height with the slant height of a pyramid. When given the height and length of one side, some students struggle with knowing how to apply the Pythagorean Theorem to solve for the slant height. These students may benefit from holding a three-dimensional solid and viewing the slant height, base, and side length. It may also help students to draw the right triangle inside the figure so they know which side represents a leg and a hypotenuse. *The answer is: 136 cm^2.*

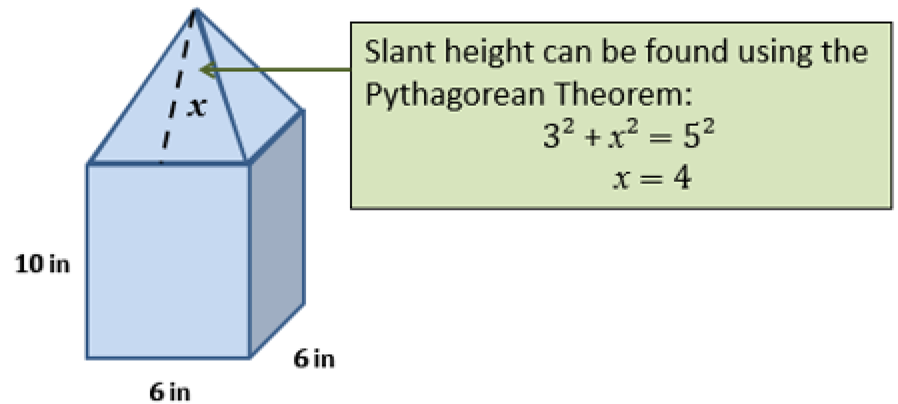
Mathematical Reasoning: Students need additional practice determining surface area and volume of three-dimensional composite figures.

A statue consists of a square-based pyramid and a rectangular prism with congruent bases. The square-based pyramid has a perpendicular height of 5 inches. Specific measurements of the statue are shown in this figure.



What is the total surface area of the statue represented by this composite figure? [Figure is not drawn to scale.]

Some students will find the sum of the surface areas of the square-based pyramid and the rectangular prism without removing the shared base from each of the formulas. Other students will have difficulty determining the slant height of the figure. [Refer to the image.] *The answer is 324 square inches.*



Concepts and Connections

CONCEPTS

Analyzing and describing geometric objects, the relationships and structures among them, or the space that they occupy can be used to classify, quantify, measure, or count one or more attributes.

CONNECTIONS

- *Within the grade level/course*:
  - G.RLT.2 – The student will analyze, prove, and justify the relationships of parallel lines cut by a transversal.
  - G.PC.3 – The student will solve problems, including those in context, by applying properties of circles.
  - G.DF.2 – The student will determine the effect of changing one or more dimensions of a three-dimensional geometric figure and describe the relationship between the original and changed figure.
- *Vertical Progression*:
  - 8.MG.2 – The student will investigate and determine the surface area of square-based pyramids and the volume of cones and square-based pyramids.
  - 8.MG.5 – The student will solve area and perimeter problems involving composite plane figures, including those in context.

- A.EO.2 – The student will perform operations on and factor polynomial expressions in one variable.
- A.EI.1 – The student will represent, solve, explain, and interpret the solution to multistep linear equations and inequalities in one variable and literal equations for a specified variable.
- A.EI.3 – The student will represent, solve, and interpret the solution to a quadratic equation in one variable.

Resources to Support Local Curriculum

- A list of approved textbooks and instructional materials is posted on the VDOE website.
- EOC Geometry Formula Sheet (PDF)
- Surface Area and Volume Exemplar Mathematics Instructional Plan (Word | PDF)

G.DF.2

The student will determine the effect of changing one or more dimensions of a three-dimensional geometric figure and describe the relationship between the original and changed figure.

Students will demonstrate the following Knowledge and Skills:

a) Describe how changes in one or more dimensions of a figure affect other derived measures (perimeter, area, total surface area, and volume) of the figure.

b) Describe how changes in surface area and/or volume of a figure affect the measures of one or more dimensions of the figure.

c) Solve problems, including those in context, involving changing the dimensions or derived measures of a three-dimensional figure.

d) Compare ratios between side lengths, perimeters, areas, and volumes of similar figures.

e) Recognize when two- and three-dimensional figures are similar and solve problems, including those in context, involving attributes of similar geometric figures.

Understanding the Standard

- A change in one dimension of a figure results in a predictable change in area. The resulting figure may or may not be similar to the original figure.
- A change in one dimension of a figure results in a predictable change in volume. The resulting figure may or may not be similar to the original figure.
- A change in one dimension of a figure results in a predictable change in perimeter. The resulting figure may or may not be similar to the original figure.
- A change in surface area and/or volume results in a predictable change in one or more dimensions of the figure. The resulting figure may or may not be similar to the original figure.
- A constant ratio, the scale factor, exists between corresponding dimensions of similar figures.
- If the ratio between dimensions of similar figures is *a*:*b* then:
    - the ratio of their areas is $a^2 : b^2$.
    - the ratio of their volumes is $a^3 : b^3$.
- Proportional reasoning is important when comparing attribute measures in similar figures.

Skills in Practice

While the five process goals are expected to be embedded in each standard, the Skills in Practice highlight the most prevalent process goals in relation to the content presented.

Mathematical Problem Solving: Students need practice determining the relationship between changes that affect one dimension (linear), changes that affect two dimensions (area), and changes that affect three dimensions (volume), particularly when figures are not provided.

- A cube has a volume of 36 cm^3. If all dimensions of the original cube were tripled, what is the volume of the new cube?

  Some students will not cube the changes in dimension when all three dimensions are impacted. Students may also misunderstand "triple the value" and cube the values of the original dimensions. It may be helpful to have students list the length, width, and height of the cube, multiply each by three, and then find the volume. *The answer to this problem is 972 cubic cm.*

- Cube A is similar to Cube B. The ratio of the volumes of the two cubes is 125:512. Find the ratio of their surface areas.

  A common misconception that some students may have is that the surface areas of similar solids change at the same rate as either the volume or the sides and perimeters. This may indicate that students do not understand the linear ratio must be found first. Students with this misconception would benefit by exploring how the relationships between the sides, perimeters, areas and volumes change when comparing two cubes.

Mathematical Connections: Students need practice determining the relationship between changes that affect one dimension (linear), changes that affect two dimensions (area), and changes that affect three dimensions (volume), particularly when figures are not provided. Teachers can incorporate geometric objects into the learning experience to include various sized cans and nets.

- A right-circular cylinder has a surface area of 96 square inches. Each dimension of this cylinder is multiplied by $\frac{1}{2}$ to create a new cylinder. What is the surface area of the new cylinder?

  The answer is 24 sq. in.

Concepts and Connections

CONCEPTS

Analyzing and describing geometric objects, the relationships and structures among them, or the space that they occupy can be used to classify, quantify, measure, or count one or more attributes.

Connections

- *Within the grade level/course*:
  - G.TR.3 – The student will, given information in the form of a figure or statement, prove and justify two triangles are similar using direct and indirect proofs, and solve problems, including those in context, involving measured attributes of similar triangles.
  - G.PC.1 – The student will prove and justify theorems and properties of quadrilaterals, and verify and use properties of quadrilaterals to solve problems, including the relationships between the sides, angles, and diagonals.
  - G.PC.3 – The student will solve problems, including those in context, by applying properties of circles.
  - G.DF.1 – The student will create models and solve problems, including those in context, involving surface area and volume of rectangular and triangular prisms, cylinders, cones, pyramids, and spheres.
- *Vertical Progression*:
  - 6.NS.3 – The student will recognize and represent patterns with whole number exponents and perfect squares.
  - 7.MG.2 – The student will solve problems and justify relationships of similarity using proportional reasoning.
  - 8.MG.2 – The student will investigate and determine the surface area of square-based pyramids and the volume of cones and square-based pyramids.
  - 8.MG.5 – The student will solve area and perimeter problems involving composite plane figures, including those in context.
  - A.EO.4 – The student will simplify and determine equivalent radical expressions involving square roots of whole numbers and cube roots of integers.
  - A.EI.1 – The student will represent, solve, explain, and interpret the solution to multistep linear equations and inequalities in one variable and literal equations for a specified variable.
  - A.EI.3 – The student will represent, solve, and interpret the solution to a quadratic equation in one variable.

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